

# Hopf bifurcation with zero frequency and imperfect $SO(2)$ symmetry

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## Abstract

Rotating waves are periodic solutions in  $SO(2)$  equivariant dynamical systems. Their precession frequency changes with parameters and it may change sign, passing through zero. When this happens, the dynamical system is very sensitive to imperfections that break the  $SO(2)$  symmetry and the waves may become trapped by the imperfections, resulting in steady solutions that exist in a finite region in parameter space. This is the so-called pinning phenomenon. In this study, we analyze the breaking of the  $SO(2)$  symmetry in a dynamical system close to a Hopf bifurcation whose frequency changes sign along a curve in parameter space. The problem is very complex, as it involves the complete unfolding of high codimension. A detailed analysis of different types of imperfections indicates that a pinning region surrounded by infinite-period bifurcation curves appears in all cases. Complex bifurcational processes, strongly dependent on the specifics of the symmetry breaking, appear very close to the intersection of the Hopf bifurcation and the pinning region. Scaling laws of the pinning region width, and partial breaking of  $SO(2)$  to  $Z_m$ , are also considered. Previous and new experimental and numerical studies of pinned rotating waves are reviewed in light of the new theoretical results.

## 1 Introduction

Dynamical systems theory plays an important role in many areas of mathematics and physics because it provides the building blocks that allow us to understand the changes many physical systems experience in their dynamics when parameters are varied. These building blocks are the generic bifurcations (saddle-node, Hopf, etc.) that any arbitrary physical system experiences under parameter variation, regardless of the physical mechanisms underlying the dynamics. When one single parameter of the system under consideration is varied, codimension-one bifurcations are expected. If the system depends on more parameters, higher codimension bifurcations appear and they act as organizing centers of the dynamics.

The presence of symmetries changes the nature and type of bifurcations that a dynamical system may undergo. Symmetries play an important role in many idealized situations, where simplifying assumptions and the consideration of simple geometries result in dynamical systems equivariant under a certain symmetry group. Bifurcations with symmetry have been widely studied (Golubitsky & Schaeffer, 1985; Golubitsky *et al.*, 1988; Chossat & Iooss, 1994; Golubitsky & Stewart, 2002; Chossat & Lauterbach, 2000; Crawford & Knobloch, 1991). However, in any real system, the symmetries are only approximately fulfilled, and the breaking of the symmetries, due to the presence of noise, imperfections and/or other phenomena, is always present. There are numerous studies of how imperfect symmetries lead to dynamics that are unexpected in the symmetric problem, e.g.

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(Keener, 1987; Campbell & Holmes, 1992; Knobloch *et al.*, 1995; Hirschberg & Knobloch, 1996; Dangelmayr *et al.*, 1997; Lamb & Wulff, 2000). However, a complete theory is currently unavailable.

One observed consequence of imperfections in systems that support propagating waves is that the waves may become trapped by the imperfections (e.g., see Keener, 1987; Westerburg & Busse, 2003; Thiele & Knobloch, 2006*a,b*). In these various examples, the propagation direction is typically biased. However, a more recent problem has considered a case where a rotating wave whose sense of precession changes sign is pinned by symmetry-breaking imperfections (Abshagen *et al.*, 2008). We are unaware of any systematic analysis of the associated normal form dynamics for such a problem and this motivates the present study.

When a system is invariant to rotations about an axis (invariance under the  $SO(2)$  symmetry group),  $SO(2)$ -symmetry-breaking Hopf bifurcations result in rotating waves, consisting of a pattern that rotates about the symmetry axis at a given precession frequency without changing shape. This frequency is parameter dependent, and in many problems, when parameters are varied, the precession frequency changes sign along a curve in parameter space. What has been observed in different systems is that in the presence of imperfections, the curve of zero frequency becomes a band of finite width in parameter space. Within this band, the rotating wave becomes a steady solution. This is the so-called pinning phenomenon. It can be understood as the attachment of the rotating pattern to some stationary imperfection of the system, so that the pattern becomes steady, as long as its frequency is small enough so that the imperfection is able to stop the rotation. This pinning phenomenon bears some resemblance to the frequency locking phenomena, although in the frequency locking case we are dealing with a system with two non-zero frequencies and their ratio becomes constant in a region of parameter space (a resonance horn), whereas here we are dealing with a single frequency crossing zero.

In the present paper, we analyze the breaking of  $SO(2)$  symmetry in a dynamical system close to a Hopf bifurcation whose frequency changes sign along a curve in parameter space. The analysis shows that breaking  $SO(2)$  symmetry is much more complex than expected, resulting in a bifurcation of high codimension (about nine). Although it is not possible to analyze in detail such a complex and high-codimension bifurcation, we present here the analysis of five different ways to break  $SO(2)$  symmetry. This is done by introducing into the normal form all the possible terms, up to and including second order, that break the symmetry, and analyzing each of these five terms separately. Three of these particular cases have already been analyzed in completely different contexts unrelated to the pinning phenomenon (Gambaudo, 1985; Wagener, 2001; Broer *et al.*, 2008; Saleh & Wagener, 2010). In the present study, we extract the common features that are associated with the pinning. In all cases, we find that a band of pinning solutions appears around the zero frequency curve from the symmetric case, and that the band is delimited by curves of infinite-period bifurcations. The details of what happens when the infinite-period bifurcation curves approach the Hopf bifurcation curve are different in the five cases, and involve complicated dynamics with several codimension-two bifurcations occurring in a small region of parameter space as well as several global bifurcations.

Interest in the present analysis is two-fold. First of all, although the details of the bifurcational process close to the zero-frequency Hopf point are very complicated and differ from case to case, for all cases we observe the appearance of a pinning band delimited by infinite-period bifurcations of homoclinic type that, away from the small region of complicated dynamics, are SNIC bifurcations (saddle-node on an invariant circle bifurcation, e.g. see Strogatz, 1994). Secondly, some of the scenarios analyzed are important *per se*, as they correspond to the generic analysis of a partial breaking of the  $SO(2)$  symmetry, so that after the introduction of perturbations, the system still retains a discrete symmetry (the  $Z_2$  case is analyzed in detail).

The paper is organized as follows. In section §2 the properties of a Hopf bifurcation with  $SO(2)$  symmetry with the precession frequency crossing through zero are summarized, and the general unfolding of the  $SO(2)$  symmetry breaking process is discussed. The next sections explore the particulars of breaking the symmetry at order zero (§3), one (§4) and two (§5). Sections §4 and §5.1 are particularly interesting because they consider the symmetry-breaking processes  $SO(2) \rightarrow Z_2$  and  $SO(2) \rightarrow Z_3$  which are readily realized experimentally. Section §7 extracts the general features of the pinning problem from the analysis of the specific cases carried out in the earlier sections.

Section §8 presents comparisons with experiments and numerical computations in two real problems in fluid dynamics, illustrating the application of the general theory developed in the present study. Finally, in §9, conclusions and perspectives are presented.

## 2 Hopf bifurcation with $SO(2)$ symmetry and zero frequency

The normal form for a Hopf bifurcation is

$$\dot{z} = z(\mu + i\omega - c|z|^2), \quad (1)$$

where  $z$  is the complex amplitude of the bifurcating periodic solution,  $\mu$  is the bifurcation parameter, and  $\omega$  and  $c$  are functions of  $\mu$  and generically at the bifurcation point ( $\mu = 0$ ) both are different from zero. It is the non-zero character of  $\omega$  that allows one to eliminate the quadratic terms in  $z$  in the normal form. This is because the normal form  $\dot{z} = P(z, \bar{z})$  satisfies (e.g., see Haragus & Iooss, 2011)

$$P(e^{-i\omega t}z, e^{i\omega t}\bar{z}) = e^{-i\omega t}P(z, \bar{z}), \quad (2)$$

where  $P$  is a low order polynomial that captures the dynamics in a neighborhood of the bifurcation point. If  $\omega = 0$ , this equation becomes an identity and  $P$  cannot be simplified. The case  $\omega = 0$  is a complicated bifurcation and it depends on the details of the double-zero eigenvalue of the linear part  $L$  of  $P$ ; as  $z = x + iy$  is complex, the matrix of  $L$  using the real coordinates  $(x, y)$  is a real  $2 \times 2$  matrix. If  $L$  is not completely degenerate, that is

$$L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad (3)$$

then we have the well-studied Takens–Bogdanov bifurcation, whereas the completely degenerate case,

$$L = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (4)$$

is a high-codimension bifurcation that has not been completely analyzed.

If the system has  $SO(2)$  symmetry, it must also satisfy

$$P(e^{im\theta}z, e^{-im\theta}\bar{z}) = e^{im\theta}P(z, \bar{z}), \quad (5)$$

where  $Z_m$  is the discrete symmetry retained by the bifurcated solution. When the group  $Z_m$  is generated by rotations of angle  $2\pi/m$  about an axis of  $m$ -fold symmetry, as is usually the case with  $SO(2)$ , then the group is also called  $C_m$ . Equations (2) and (5) are completely equivalent and have the same implications for the normal form structure. Advancing in time is the same as rotating the solution by a certain angle ( $\omega t = m\theta$ ); the bifurcated solution is a rotating wave. Therefore, if  $\omega$  becomes zero by varying a second parameter, we still have the same normal form (1), due to (5), with  $\omega$  replaced by a small parameter  $\nu$ :

$$\dot{z} = z(\mu + i\nu - c|z|^2). \quad (6)$$

The Hopf bifurcation with  $SO(2)$  symmetry and zero frequency is, in this sense, trivial. Introducing the modulus and phase of the complex amplitude  $z = re^{i\phi}$ , the normal form becomes

$$\begin{aligned} \dot{r} &= r(\mu - ar^2), \\ \dot{\phi} &= \nu - br^2, \end{aligned} \quad (7)$$

where  $c = a + ib$ , and let us assume for the moment that  $a$  and  $b$  are positive. The bifurcation frequency in (7) is now the small parameter  $\nu$ . The bifurcated solution  $RW_m$  exists only for  $\mu > 0$ , and has amplitude  $r = \sqrt{\mu/a}$  and frequency  $\omega = \nu - b\mu/a$ . The limit cycle  $RW_m$  becomes an invariant set of steady solutions along the straight line  $\mu = a\nu/b$  (labeled L in figure 1) where

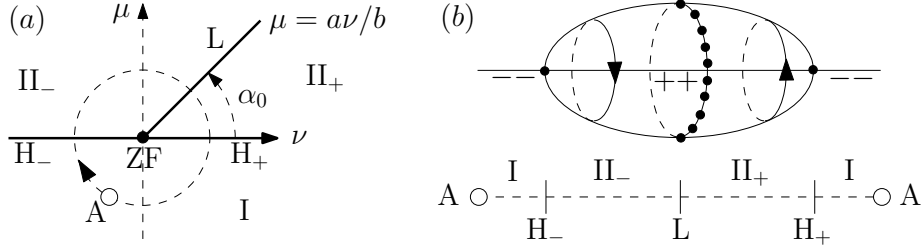


Figure 1: Hopf bifurcation with  $SO(2)$  symmetry and zero frequency; part (a) shows the bifurcation diagram, where the thick lines are bifurcation curves, and part (b) shows the bifurcations along the path A shown in (a). The fixed point curve is labeled with the signs of its eigenvalues. In regions  $II_-$  and  $II_+$  the limit cycles, born at the Hopf bifurcations  $H_-$  and  $H_+$ , rotate in opposite senses. L is the line where the limit cycle becomes an invariant curve of fixed points.

the frequency of  $RW_m$  goes to zero; the angle between L and the Hopf bifurcation curve (the horizontal axis  $\mu = 0$ ) is  $\alpha_0$ . The bifurcation diagram and a schematic of the bifurcations along a one-dimensional path is also shown in figure 1. The bifurcation point  $\mu = \nu = 0$ , labeled ZF (zero-frequency Hopf point) in figure 1(a), is a codimension-two bifurcation. It coincides with the generic Hopf bifurcation, except that it includes a line L along which the bifurcated solution has zero frequency.

Assuming  $c \neq 0$ , we can simplify (7) by scaling  $z$  so that  $|c| = 1$ ; we will write

$$c = a + ib = ie^{-i\alpha_0} = \sin \alpha_0 + i \cos \alpha_0, \quad b + ia = e^{i\alpha_0}, \quad (8)$$

which helps simplify subsequent expressions. The case  $a$  and  $b$  both positive, which we will analyze in detail in the following sections, corresponds to one of the fluid dynamics problems that motivated the present analysis (see Abshagen *et al.*, 2008; Pacheco *et al.*, 2011, and §88.1). For other signs of  $a$  and  $b$ , analogous conclusions can be drawn. It is of particular interest to consider the subcritical case  $a < 0$  as it corresponds to the other fluid dynamics problem analyzed here (see Marques *et al.*, 2007; Lopez & Marques, 2009, and §88.2). By reversing time and changing the sign of  $\mu$  and  $\nu$ , we obtain exactly the same normal form (7) but with the opposite sign of  $a$  and  $b$ . By changing the sign of  $\phi$  and  $\nu$ , we obtain (7) with the opposite sign of  $b$ . Therefore, all possible cases corresponding to different signs of  $a$  and  $b$  can be reduced to the case where  $a$  and  $b$  are both positive.

## 2.1 Unfolding the Hopf bifurcation with zero frequency

If the  $SO(2)$  symmetry in the normal form (6) is completely broken, and no symmetry remains, then the restrictions imposed on the normal form by (5) disappear completely and all the terms in  $z$  and  $\bar{z}$  missing from (6) will reappear multiplied by small parameters. This means that the normal form will be

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \epsilon_1 + \epsilon_2 \bar{z} + \epsilon_3 \bar{z}^2 + \epsilon_4 z \bar{z} + \epsilon_5 z^2, \quad (9)$$

where additional cubic terms have been neglected because we assume  $c \neq 0$  and that  $cz|z|^2$  will be dominant. As the  $\epsilon_i$  are complex, we have a problem with 12 parameters. Additional simplifications can be made in order to obtain the so-called hypernormal form; this method is extensively used by Kuznetsov (2004), for example. Unfortunately, many of the simplifications rely on having some low-order term in the normal form being non-zero with a coefficient of order one. For example, if  $\omega \neq 0$ , it is possible to make  $c$  real by using a time re-parametrization. In our problem, all terms up to and including second order are zero or have a small coefficient, and so only a few simplifications are possible. These simplifications are an infinitesimal translation of  $z$  (two parameters), and an arbitrary shift in the phase of  $z$  (one parameter). Using these transformations the twelve parameters can be reduced to nine. In particular, one of either  $\epsilon_4$  or  $\epsilon_5$  can be taken as zero and the other can be made real. By rescaling  $z$ , we can make  $c$  of modulus one, as in (8). A complete analysis of a normal

form depending on nine parameters, i.e. a bifurcation of codimension of about nine, is completely beyond the scope of the present paper. In the literature, only codimension-one bifurcations have been completely analyzed. Most of the codimension-two bifurcations for ODE and maps have also been analyzed, except for a few bifurcations for maps that remain outstanding (Kuznetsov, 2004). A few codimension-three and very few codimension-four bifurcations have also been analyzed (Chow *et al.*, 1994; Dumortier *et al.*, 1997), but to our knowledge, there is no systematic analysis of bifurcations of codimension greater than two.

In the following sections, we consider the five cases,  $\epsilon_1$  to  $\epsilon_5$ , separately. A combination of analytical and numerical tools allows for a detailed analysis of these bifurcations. We extract the common features of the different cases when  $\epsilon_i \ll \sqrt{\mu^2 + \nu^2}$ , which captures the relevant behavior associated with weakly breaking  $SO(2)$  symmetry. In particular, the  $\epsilon_2$  case exhibits very interesting and rich dynamics that may be present in some practical cases when the  $SO(2)$  symmetry group is not completely broken and a  $Z_2$  symmetry group, generated by the half-turn  $\theta \rightarrow \theta + \pi$ , remains.

Some general comments can be made here about these five cases, which are of the form

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \epsilon z^q \bar{z}^{p-q}, \quad (10)$$

for integers  $0 \leq q \leq p \leq 2$ , excluding the case  $p = q = 1$  which is  $SO(2)$  equivariant and so  $\epsilon$  can be absorbed into  $\mu$  and  $\nu$ . By changing the origin of the phase of  $z$ , we can modify the phase of  $\epsilon$  so that it becomes real and positive. Then, by re-scaling  $z$ , time  $t$ , and the parameters  $\mu$  and  $\nu$  as

$$(z, t, \mu, \nu) \rightarrow (\epsilon^\delta z, \epsilon^{-2\delta} t, \epsilon^{2\delta} \mu, \epsilon^{2\delta} \nu), \quad \delta = \frac{1}{3-p}, \quad (11)$$

we obtain (10) with  $\epsilon = 1$ , effectively leading to codimension-two bifurcations in each of the five cases. We expect complex behavior for  $\mu^2 + \nu^2 \lesssim \epsilon^2$ , when the three parameters are of comparable size, while the effects of small imperfections breaking  $SO(2)$  will correspond to  $\mu^2 + \nu^2 \gg \epsilon^2$ . From now on  $\epsilon = 1$  will be assumed, and we can restore the explicit  $\epsilon$ -dependence by reversing the transformation (11). Three of the five normal forms (11) have been analyzed in the literature (discussed below), focusing on the regions where  $\mu$ ,  $\nu$  and  $\epsilon$  are of comparable size; here we will also consider what happens for  $\mu^2 + \nu^2 \gg \epsilon^2$  which is particularly important for the pinning phenomenon.

The normal forms corresponding to the  $\epsilon_1$ ,  $\epsilon_2$  and  $\epsilon_3$  cases have already been analyzed in contexts completely different to the  $SO(2)$  symmetry-breaking context considered here. The context in which these problems were studied stems from low-order resonances in perturbed Hopf problems. Gambaudo (1985) studied time-periodic forcing near a Hopf bifurcation point, analyzing the problem using the Poincaré stroboscopic map. The normal forms corresponding to the 1:1, 1:2 and 1:3 strong resonances coincide with the normal forms we present below for cases with only the  $\epsilon_1$ ,  $\epsilon_2$  and  $\epsilon_3$  terms retained in (9), respectively. Later, motivated by a problem of a nonlinear oscillator with damping and quasi-periodic driving, a series of papers extended the strong resonances results of Gambaudo (1985) by studying the semi-global bifurcations for periodically and quasi-periodically perturbed driven damped oscillators near a Hopf bifurcation (see Wagener, 2001; Broer *et al.*, 2008; Saleh & Wagener, 2010, and references therein). The other two cases we consider, with only the  $\epsilon_4$  or the  $\epsilon_5$  terms retained in (9), do not appear to have been studied previously. They fall outside of the context in which the other three were studied because they do not correspond to any canonical resonance problem. We should point out that within the resonance context, the three cases studied would not make sense to consider in combination (they correspond to completely distinct frequency ratios and so would not generically occur in a single problem). In contrast, within the context motivating our study, all five cases correspond to different ways in which the  $SO(2)$  symmetry of a system may be broken, and in a physical realization, all five could co-exist. In the following sections, we present a detailed analysis of all five cases.

### 3 Symmetry breaking of $SO(2)$ with an $\epsilon$ term

The normal form to be analyzed is (10) with  $p = q = 0$  and  $\epsilon = 1$ :

$$\dot{z} = z(\mu + i\nu - c|z|^2) + 1. \quad (12)$$

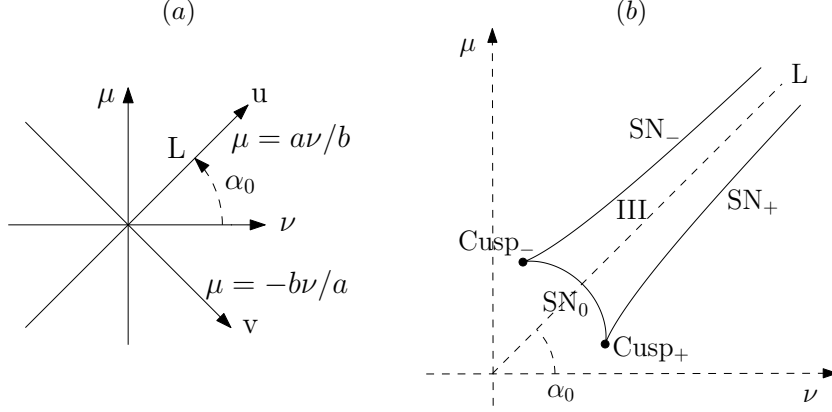


Figure 2: (a) Coordinates  $(u, v)$  in parameter space adapted to the line  $L$ , which coincides with zero frequency curve in the unperturbed  $SO(2)$  symmetric case. (b) Steady bifurcations of the fixed points corresponding to the normal form (12).  $SN_{\pm}$  and  $SN_0$  are saddle-node bifurcation curves and  $Cusp_{\pm}$  are cusp bifurcation points. In region III there exist three fixed points, and only one in the rest of parameter space.

This case has been analyzed in Gambaudo (1985); Wagener (2001); Broer *et al.* (2008); Saleh & Wagener (2010).

It is convenient to introduce coordinates  $(u, v)$  in parameter space, rotated an angle  $\alpha_0$  with respect to  $(\mu, \nu)$ , so that the line  $L$  becomes the new coordinate axis  $v = 0$ . Many features, that are symmetric with respect to  $L$  will then simplify; e.g. the distance to the bifurcation point along  $L$  is precisely  $u$ .

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} a\mu + b\nu \\ a\nu - b\mu \end{pmatrix}, \quad \begin{pmatrix} \mu \\ \nu \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \quad (13)$$

where  $a = \sin \alpha_0$ ,  $b = \cos \alpha_0$ . Figure 2(a) shows the relationship between the two coordinate systems.

### 3.1 Fixed points and their local bifurcations

The normal form (12), in terms of the modulus and phase  $z = re^{i\phi}$ , is

$$\begin{aligned} \dot{r} &= r(\mu - ar^2) + \cos \phi, \\ \dot{\phi} &= \nu - br^2 - \frac{1}{r} \sin \phi. \end{aligned} \quad (14)$$

The fixed points are given by a cubic equation in  $r^2$ , and so we do not have convenient closed forms for the corresponding roots (the Tartaglia explicit solution is extremely involved). However, it is easy to obtain the locus where two of the fixed points coalesce. The parameter space is divided into two regions, region III with three fixed points, and the rest of parameter space with one fixed point, as seen in figure 2(b). The curve separating both regions is a saddle-node curve given by (see B)

$$(u, v) = \frac{(3 + 3s^2, 2\sqrt{3}s)}{(2 + 6s^2)^{2/3}}, \quad s \in (-\infty, +\infty), \quad (15)$$

in  $(u, v)$  coordinates (13). The saddle-nodde curve is divided into three different arcs  $SN_+$ ,  $SN_-$  and  $SN_0$  by two codimension-two cusp bifurcation points,  $Cusp_{\pm}$ .  $SN_-$  corresponds to values  $s \in (-\infty, 1)$ ,  $SN_0$  to  $s \in (-1, +1)$  and  $SN_+$  to  $s \in (1, +\infty)$ . The cusp points  $Cusp_{\pm}$  have values  $s = \pm 1$ . The curves  $SN_+$  and  $SN_-$  are asymptotic to the line  $L$ , and region II is the pinning region

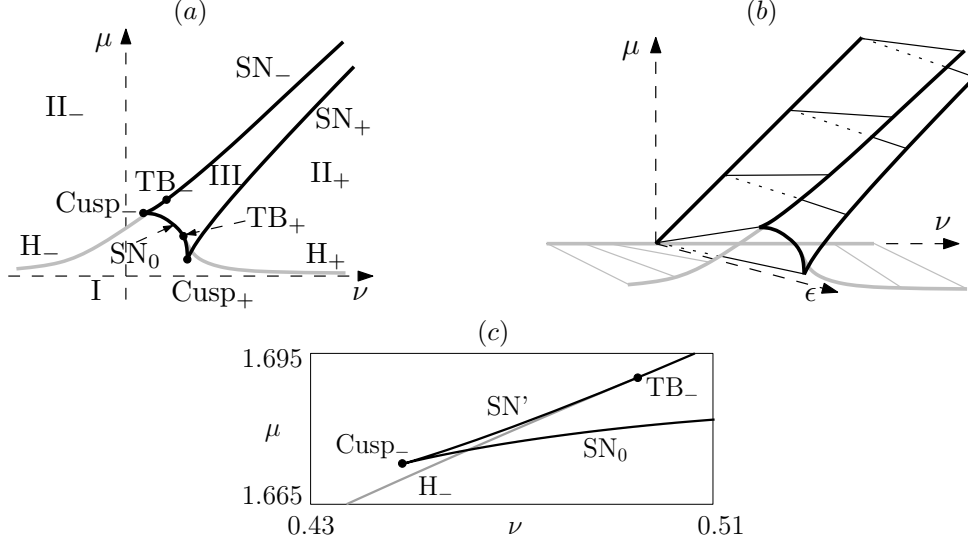


Figure 3: (a) Bifurcations of the fixed points corresponding to the normal form (12).  $SN_{\pm}$  and  $SN_0$  are saddle-node bifurcation curves,  $H_{\pm}$  are Hopf bifurcation curves,  $Cusp_{\pm}$  are cusp bifurcation points and  $TB_{\pm}$  are Takens–Bogdanov bifurcation points. A perspective view of the corresponding codimension-three bifurcation in terms of  $(\mu, \nu, \epsilon)$  is shown in (b). (c) is a zoom of (a), showing that  $Cusp_-$  and  $TB_-$  are different. The parameter range in (a) is  $(\nu, \mu) \in [-3, 6] \times [-1, 6]$ , for  $\alpha = 45^\circ$ .

in this case. The two fixed points that merge on the saddle-node curve have phase space coordinates  $z_2 = r_2 e^{i\phi_2}$ , and the third fixed point is  $z_0 = r_0 e^{i\phi_0}$ , where

$$r_0 = \left( \frac{4}{1 + 3s^2} \right)^{1/3}, \quad r_2 = \left( \frac{1 + 3s^2}{4} \right)^{1/6}, \quad (16)$$

and the phases are obtained from  $\sin \phi = r(\nu - br^2)$ ,  $\cos \phi = r(ar^2 - \mu)$ .

The Hopf bifurcations of the fixed points can be obtained by imposing the conditions  $T = 0$  and  $D > 0$ , where  $T$  and  $D$  are the trace and determinant of the Jacobian of (12). These conditions result in two curves of Hopf bifurcations (see B for details):

$$(\mu, \nu) = a^{1/3}(1 - s^2)^{1/3} \left( 2, \frac{b}{a} + \frac{s}{\sqrt{1 - s^2}} \right), \quad (17)$$

$$H_- : s \in \left( -1, -\sqrt{(1 - b)/2} \right), \quad H_+ : s \in \left( \sqrt{(1 + b)/2}, +1 \right).$$

For  $s \rightarrow \pm 1$  both curves are asymptotic to the  $\mu = 0$  axis ( $\nu \rightarrow \pm\infty$ ), the Hopf curve for  $\epsilon = 0$ ; the stable limit cycles born at these curves are termed  $C_-$  and  $C_+$  respectively. The limit cycles (rotating waves) in  $II_{\pm}$  rotate in opposite directions, and III is the pinning region where the rotation stops and we have a stable fixed point. Solutions with  $\omega = 0$ , that existed only along a single line in the absence of imperfections, now exist in a region of finite width. Figure 3(b) shows what happens when the  $\epsilon$  dependence is restored; what we have is that figure 3(a) just scales with  $\epsilon$  as indicated in (11), and the pinning region collapses onto the line L of the perfect case with  $SO(2)$  symmetry.

The other ends of the  $H_{\pm}$  curves are on the saddle-node curves of fixed points previously obtained, and at these points  $T = D = 0$ , so they are Takens–Bogdanov points  $TB_{\pm}$ , as shown in figure 3. The  $TB_-$  and  $Cusp_-$  codimension-two bifurcation points are very close, as shown in the zoomed-in figure 3(b). In fact, depending on the angle  $\alpha_0$ , the Hopf curve  $H_-$  is tangent to, and ends at, either  $SN_-$  or  $SN_0$ . For  $\alpha_0 > 60^\circ$ ,  $H_-$  ends at  $SN_0$ , and for  $\alpha_0 = 60^\circ$   $Cusp_-$  and  $TB_-$  coincide, and  $H_-$  ends at the cusp point, a very degenerate case.

From the Takens–Bogdanov points, dynamical systems theory says that two curves of homoclinic bifurcations emerge, resulting in global bifurcations around these points. Moreover, the stable limit

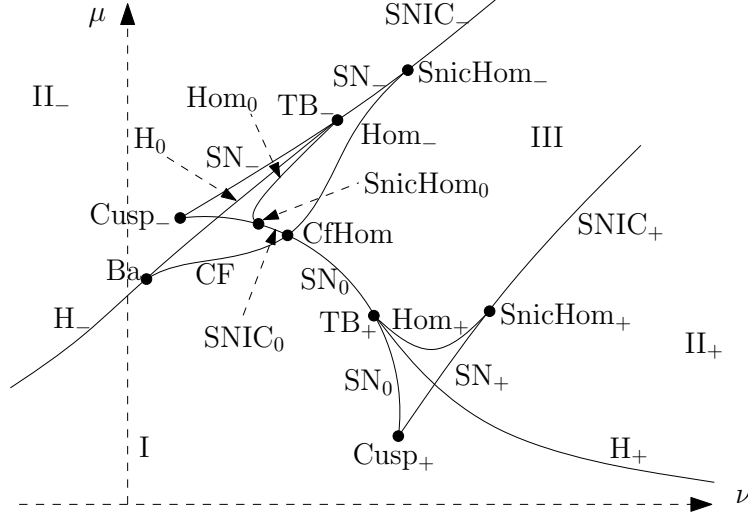


Figure 4: Schematic of the bifurcations of the normal form (12). There are seven curves of global bifurcations,  $\text{Hom}_\pm$ ,  $\text{Hom}_0$  (homoclinic collisions of a limit cycle with a saddle), CF (a cyclic-fold),  $\text{SNIC}_\pm$ , and  $\text{SNIC}_0$ , and nine codimension-two points (black circles). The regions around the codimension-two points have been enhanced for clarity.

cycles in regions  $\text{II}_\pm$  do not exist in region III, so they must disappear in additional bifurcations. These additional global bifurcations have been explored numerically and using dynamical systems theory; these are summarized in figure 4. There are nine codimension-two points organizing the dynamics of the normal form (12). Apart from the cusp and Takens–Bogdanov points already found,  $\text{Cusp}_\pm$  and  $\text{TB}_\pm$ , there are five new points: Ba, Cfhom and three different Snic-Homoclinic bifurcations,  $\text{SnicHom}_\pm$  and  $\text{SnicHom}_0$ . On  $H_-$ , before crossing the  $\text{SN}_0$  curve, the Hopf bifurcation becomes subcritical at the Bautin point Ba, and from this point a curve of cyclic-folds CF appears. This curve is the limit of the subcritical region where two periodic solutions,  $C_-$  and  $C_0$ , exist and they merge on CF.  $C_0$  is the unstable limit cycle born in the branch of  $H_-$  between  $\text{Cusp}_-$  and  $\text{TB}_-$ , from now on termed the Hopf curve  $H_0$ . Inside the pinning region these two periodic solutions disappear when they collide with a saddle fixed point along the curves  $\text{Hom}_0$  ( $C_0$  collision) and  $\text{Hom}_-$  ( $C_-$  collision), in the neighborhood of the Takens–Bogdanov point  $\text{TB}_-$ , where the homoclinic curve  $\text{Hom}_0$  is born. Away from the  $\text{TB}_-$ , for increasing values of  $\mu$  and  $\nu$ , the curve  $\text{Hom}_-$  becomes tangent to and collides with the  $\text{SN}_-$  curve, in a  $\text{SnicHom}_-$  bifurcation. Very close to  $\text{SnicHom}_-$ ,  $\text{SN}_-$  closely followed by  $\text{Hom}_-$  become indistinguishable from the  $\text{SNIC}_-$  bifurcation, in the same sense as discussed in section §4.3.

The two curves  $\text{Hom}_-$  and  $\text{Hom}_0$  on approaching  $\text{SN}_0$ , result in a couple of codimension-two bifurcation points,  $\text{SnicHom}_0$  and Cfhom. The arc of the curve  $\text{SN}_0$  between the two new points  $\text{SnicHom}_0$  and Cfhom, is a curve of saddle-node bifurcations taking place on the limit cycle  $C_0$ , resulting in the  $\text{SNIC}_0$  bifurcation curve, as shown in figure 4, and in more detail in the numerically computed inset figure 5(a).  $\text{SnicHom}_0$  is exactly the same bifurcation as  $\text{SnicHom}_\pm$ .

The cyclic fold bifurcation curve CF intersects the  $\text{SNIC}_0$  bifurcation curve past the  $\text{SnicHom}_0$  point, i.e. when the  $\text{SN}_0$  curve is a line of SNIC bifurcations, at the point Cfhom. On the  $\text{SNIC}$  curve, one of the limit cycles born at CF undergoes a SNIC bifurcation. At the point Cfhom, the SNIC bifurcation happens precisely when both limit cycles are born at CF: it is a saddle-node bifurcation of fixed points taking place on a saddle-node bifurcation of limit circles. After the Cfhom point, the  $\text{SNIC}$  curve becomes an ordinary saddle-node bifurcation curve, and there is an additional homoclinic bifurcation curve emerging from this point Cfhom,  $\text{Hom}_-$  in figure 4. The two limit cycles born at CF exist only on one side of the CF line, so when following a closed path around the Cfhom point they must disappear. One of them undergoes a SNIC bifurcation on



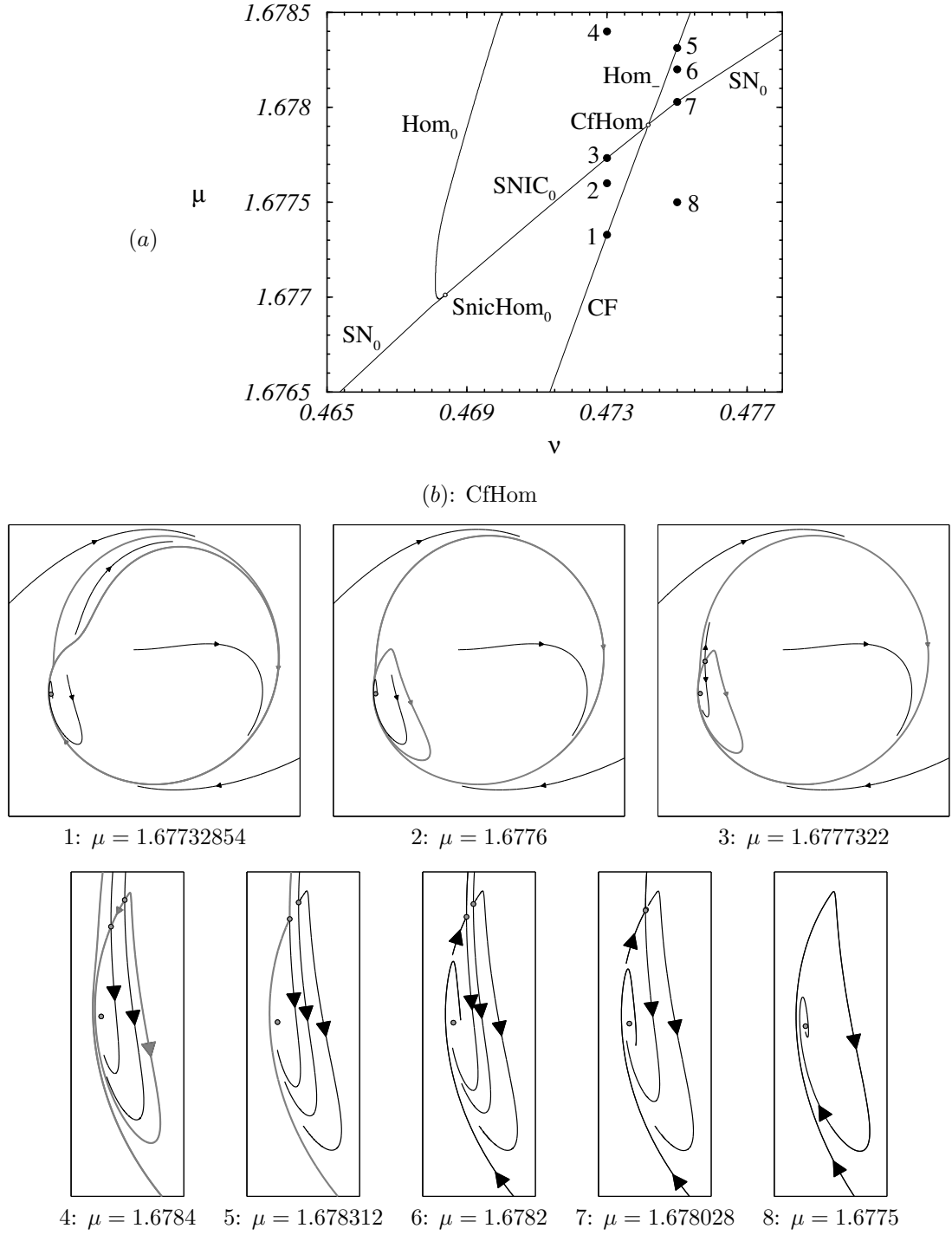


Figure 5: (a) Zoom in parameter space around the codimension-two global bifurcation points SnicHom and CfHom. (b) Phase portraits around CfHom; plots 1 to 4 at  $\nu = 0.473$ , plots 5 to 8 at  $\nu = 0.475$ , for  $\mu$  as specified.

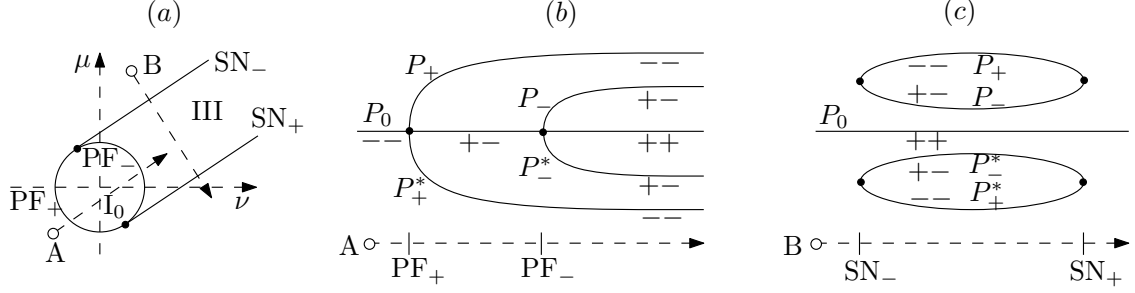


Figure 6: Steady bifurcations of fixed points corresponding to the normal form (20): (a) regions in parameter space delimited by the fixed points and their steady bifurcations; (b) and (c) bifurcations along the paths A and B shown in (a), respectively.

the SNIC curve and disappears. The other limit cycle collides with the saddle point born at the saddle-node curve  $SN_0$  (or  $SNIC_0$ ) and disappears at the homoclinic collision  $Hom_-$ . Numerically computed phase portraits illustrating these processes around the CffHom point are shown in figure 5.

The other Takens–Bogdanov point  $TB_+$  does not present any additional complications. The homoclinic curve  $Hom_+$  emerging from it approaches and intersect the  $SN_+$  curve in a  $SnicHom_+$  codimension two point, as shown in figure 4. For large values of  $\mu^2 + \nu^2$  the stable limit cycles in regions  $II_{\pm}$  disappear at  $SNIC_{\pm}$  (saddle-node on an invariant circle) bifurcation curves. On these curves, a saddle-node bifurcation of fixed points takes place on top of the limit cycle, and the cycle disappears in an infinite-period bifurcation. What remains, and is observable, is the stable fixed point born at the saddle-node.

The width of the pinning  $w(d, \epsilon)$  region away from the origin (large  $s$ ) is easy to compute from (15):

$$d = u \sim (3s^2/4)^{1/3}, \quad w = 2v \sim (16/\sqrt{3}s)^{1/3} \Rightarrow w = 2/\sqrt{d}. \quad (18)$$

Restoring the  $\epsilon$  dependence, we obtain  $w(d, \epsilon) = 2\epsilon/\sqrt{d}$ . The pinning region becomes narrower away from the bifurcation point, and the width is proportional to  $\epsilon$ , the size of the imperfection.

## 4 Symmetry breaking of $SO(2)$ to $Z_2$ , with an $\epsilon\bar{z}$ term

The  $\epsilon_2\bar{z}$  term in (9) corresponds to breaking  $SO(2)$  symmetry in a way that leaves a system with  $Z_2$  symmetry, corresponding to invariance under a half turn. The normal form to be analyzed is (10) with  $p = 1$ ,  $q = 0$  and  $\epsilon = 1$ :

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \bar{z}. \quad (19)$$

The new normal form (19) is still invariant to  $z \rightarrow -z$ , or equivalently, the half-turn  $\phi \rightarrow \phi + \pi$ . This is all that remains of the  $SO(2)$  symmetry group, which is reduced to  $Z_2$ , generated by the half-turn. In fact, the  $Z_2$  symmetry implies that  $P(z, \bar{z})$  in  $\dot{z} = P(z, \bar{z})$  must be odd:  $P(-z, -\bar{z}) = -P(z, \bar{z})$ , which is (5) for  $\theta = \pi$ , the half turn. Therefore, (19) is the unfolding corresponding to the symmetry breaking of  $SO(2)$  to  $Z_2$ . This case has also been analyzed in Gambaudo (1985); Wagener (2001); Broer *et al.* (2008); Saleh & Wagener (2010).

Writing the normal form (19) in terms of the modulus and phase of  $z = re^{i\phi}$  gives

$$\begin{aligned} \dot{r} &= r(\mu - ar^2) + r \cos 2\phi, \\ \dot{\phi} &= \nu - br^2 - \sin 2\phi. \end{aligned} \quad (20)$$

### 4.1 Fixed points and their bifurcations

The normal form (19), or (20), admits up to five fixed points. One is the origin  $r = 0$ , the trivial solution  $P_0$ . The other fixed points come in two pairs of  $Z_2$ -symmetric points: one is the

pair  $P_+ = r_+ e^{i\phi_+}$  and  $P_+^* = -r_+ e^{i\phi_+}$ , and the other pair is  $P_- = r_- e^{i\phi_-}$  and  $P_-^* = -r_- e^{i\phi_-}$ . Coefficients  $r_\pm$  and  $\phi_\pm$  are given by

$$r_\pm^2 = a\mu + b\nu \pm \Delta, \quad \phi_\pm = (\alpha_0 \pm \alpha_1)/2, \quad (21)$$

$$\Delta^2 = 1 - (a\nu - b\mu)^2, \quad e^{i\alpha_1} = a\nu - b\mu - i\Delta. \quad (22)$$

The details of the computations are given in C. There are three different regions in the  $(\mu, \nu)$ -parameter plane: region III, where there exist five fixed points,  $P_0$ ,  $P_\pm$  and  $P_\pm^*$ ; region I<sub>0</sub> where three fixed points exist,  $P_0$ ,  $P_+$  and  $P_+^*$ ; and the rest of the parameter space where only  $P_0$  exists. These three regions are separated by four curves along which steady bifurcations between the different fixed points take place, as shown in figure 6. Along the semicircle

$$\text{PF}_+ : \mu^2 + \nu^2 = 1 \text{ and } a\mu + b\nu < 0, \quad (23)$$

the two symmetrically-related solutions  $P_+$  and  $P_+^*$  are born in a pitchfork bifurcation of the trivial branch  $P_0$ . Along the semicircle

$$\text{PF}_- : \mu^2 + \nu^2 = 1 \text{ and } a\mu + b\nu > 0, \quad (24)$$

the two symmetrically-related solutions  $P_-$  and  $P_-^*$  are born in a pitchfork bifurcation of the trivial branch  $P_0$ . Along the two half-lines

$$\text{SN}_+ : \mu = (a\nu - 1)/b, \quad \text{SN}_- : \mu = (a\nu + 1)/b, \quad \text{both with } a\mu + b\nu > 0, \quad (25)$$

a saddle-node bifurcation takes place. It is a double saddle-node, due to the  $Z_2$  symmetry; we have one saddle-node involving  $P_+$  and  $P_-$ , and the  $Z_2$ -symmetric saddle-node between  $P_+^*$  and  $P_-^*$ . Bifurcation diagrams along the paths A and B in figure 6(a) are shown in parts (b) and (c) of the same figure.

We can compare with the original problem with  $SO(2)$  symmetry, corresponding to  $\epsilon = 0$ . In order to do that, the  $\epsilon$  dependence will be restored in this paragraph. The single line L ( $\mu = \nu \tan \alpha_0$ ) where  $\omega = 0$  and nontrivial fixed points exist in the perfect problem, becomes a region of width  $2\epsilon$  in the imperfect problem, where up to four fixed points exist, in addition to the base state  $P_0$ ; they are the remnants of the circle of fixed points in the original problem. Solutions with  $\omega = 0$ , that existed only along a single line in the absence of imperfections, now exist in a region bounded by the semicircle  $F_+$  and the half-lines  $a\nu - b\mu = \pm\epsilon$ ; this region will be termed the pinning region. It bears some relationship with the frequency-locking regions appearing in Neimark-Sacker bifurcations, in the sense that here we also have frequency locking, but with  $\omega = 0$ . The width of the pinning region is proportional to  $\epsilon$ , a measure of the breaking of  $SO(2)$  symmetry due to imperfections.

In the absence of imperfections ( $\epsilon = 0$ ) the  $P_0$  branch loses stability to a Hopf bifurcation along the curve  $\mu = 0$ . Let us analyze the stability of  $P_0$  in the imperfect problem. Using Cartesian coordinates  $z = x + iy$  in (19) we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu + 1 & -\nu \\ \nu & \mu - 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - (x^2 + y^2) \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}. \quad (26)$$

The eigenvalues of  $P_0$  are the eigenvalues of the linear part of (26),  $\lambda_\pm = \mu \pm \sqrt{1 - \nu^2}$ . There is a Hopf bifurcation ( $\Im \lambda_\pm \neq 0$ ) when  $\mu = 0$  and  $|\nu| > 1$ , i.e. on the line  $\mu = 0$  outside region II; the Hopf frequency is  $\omega = \text{sign}(\nu)\sqrt{\nu^2 - 1}$ . The sign of  $\omega$  is the same as the sign of  $\nu$ , from the  $\phi$  equation in (20). Therefore, we have a Hopf bifurcation with positive frequency along  $H_+$  ( $\mu = 0$  and  $\nu > 1$ ) and a Hopf bifurcation with negative frequency along  $H_-$  ( $\mu = 0$  and  $\nu < -1$ ). The bifurcated periodic solutions are stable limit cycles  $C_+$  and  $C_-$ , respectively.

The Hopf bifurcations of the  $P_\pm$  and  $P_\pm^*$  points can be studied analogously. The eigenvalues of the Jacobian of the right-hand side of (26) at a fixed point characterize the different bifurcations that the fixed point can undergo. Let  $T$  and  $D$  be the trace and determinant of  $J$ . The eigenvalues are given by

$$\lambda^2 - T\lambda + D = 0 \quad \Rightarrow \quad \lambda = \frac{1}{2}(T \pm \sqrt{Q}), \quad Q = T^2 - 4D. \quad (27)$$

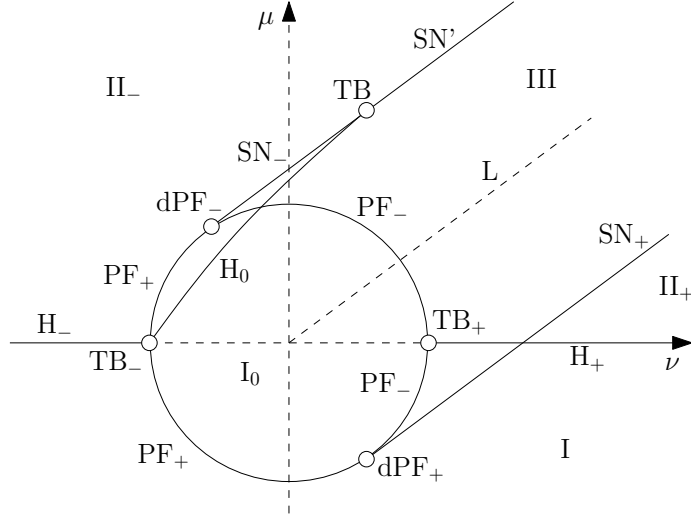


Figure 7: Local bifurcations of fixed points in the symmetry breaking of  $SO(2)$  to  $Z_2$  case. Codimension-one bifurcation curves: Hopf curves  $H_{\pm}$  and  $H_0$ , pitchforks  $PF_{\pm}$ , and saddle-nodes  $SN_{\pm}$ . Codimension-two bifurcation points: degenerate pitchforks  $dPF_{\pm}$ , Takens–Bogdanov  $TB_{\pm}$  and  $TB_{\pm}$ .  $L$  is the zero-frequency curve in the  $SO(2)$  symmetric case.

A Hopf bifurcation takes place for  $T = 0$  and  $Q < 0$ . The equation  $T = 0$  at the four points  $P_{\pm}$  results in the ellipse  $\mu^2 - 4ab\mu\nu + 4a^2\nu^2 = 4a^2$  (see C for details). This ellipse is tangent to  $SN_-$  at the point  $(\mu, \nu) = (2b, (b^2 - a^2)/a)$ . The condition  $Q < 0$  is only satisfied by  $P_+$  and  $P_+^*$  on the elliptic arc  $H_0$  from  $(\mu, \nu) = (0, -1)$  to  $(2b, (b^2 - a^2)/a)$ :

$$\mu = 2ab\nu + 2a\sqrt{1 - a^2\nu^2}, \quad \nu \in [-1, (b^2 - a^2)/a]. \quad (28)$$

The elliptic arc  $H_0$  is shown in figure 7. Along this arc a pair of unstable limit cycles  $C_0$  and  $C_0^*$  are born around the fixed points  $P_+$  and  $P_+^*$ , respectively.

## 4.2 Codimension-two points

The local codimension-one bifurcations of the fixed points are now completely characterized. There are two curves of saddle-node bifurcations, two curves of pitchfork bifurcations and three Hopf bifurcation curves. These curves meet at five codimension-two points. The analysis of the eigenvalues at these points, and of the symmetry of the bifurcating points ( $P_0$ ,  $P_+$  and  $P_+^*$ ), characterizes these points as two degenerate pitchforks  $dPF_{\pm}$ , two Takens–Bogdanov bifurcations with  $Z_2$  symmetry  $TB_{\pm}$ , and a double Takens–Bogdanov bifurcation  $TB$ , as shown in figure 7.

The degenerate pitchforks  $dPF_{\pm}$  correspond to the transition between supercritical and subcritical pitchfork bifurcations. At these points, a saddle-node curve is born, and only fixed points are involved in the neighboring dynamics. The only difference between  $dPF_+$  and  $dPF_-$  is the stability of the base state  $P_0$ ; it is stable outside the circle  $\mu^2 + \nu^2 = 1$  at  $dPF_+$  and unstable at  $dPF_-$ . Schematics of the bifurcations along a one-dimensional path in parameter space around the  $dPF_+$  and  $dPF_-$  points are illustrated in figure 8. The main difference, apart from the different stability properties of  $P_0$ ,  $P_+$  and  $P_+^*$ , is the existence of the limit cycle  $C_-$  surrounding the three fixed points in case (b),  $dPF_-$ .

The Takens–Bogdanov bifurcation with  $Z_2$  symmetry has two different scenarios (Chow *et al.*, 1994), and they differ in whether one or two Hopf curves emerge from the bifurcation point. In our problem, bifurcation point  $TB_+$  has a single Hopf curve,  $H_+$ , while the  $TB_-$  point has two Hopf curves,  $H_-$  and  $H_0$ , emerging from the bifurcation point. The scenario  $TB_+$  is depicted in figure 9, showing the bifurcation diagram as well as the bifurcations along a closed one-dimensional path around the codimension-two point. We have also included the  $P_+$  and  $P_+^*$  solutions that merge with

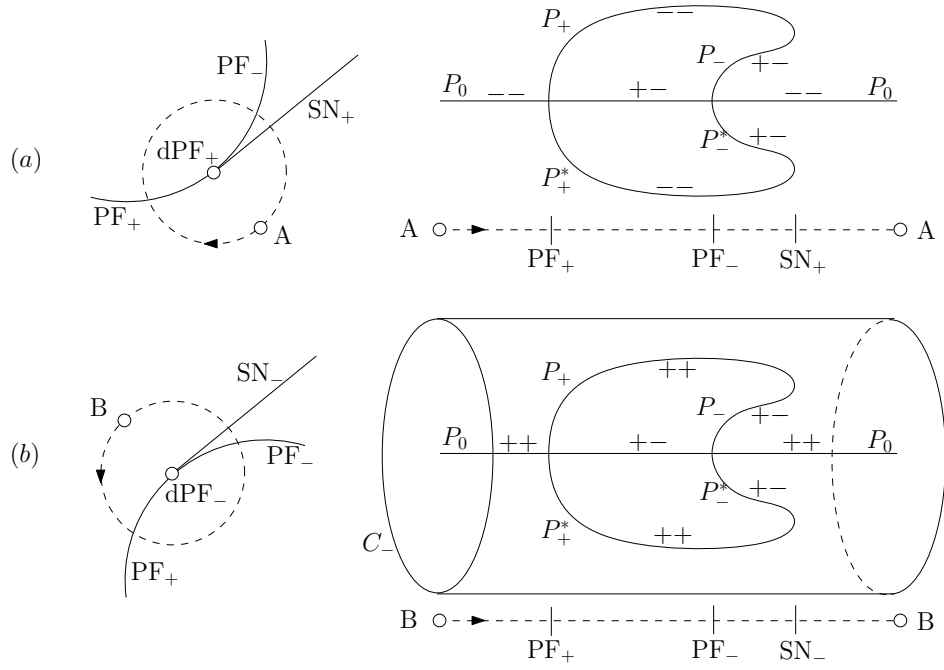


Figure 8: Schematics of the degenerate pitchfork bifurcations (a)  $dPF_+$  and (b)  $dPF_-$ . On the left, bifurcation curves emanating from  $dPF_{\pm}$  in parameter space are shown, along with a closed one-dimensional path (dashed); shown on the right are schematics of the bifurcations along the closed path, starting and ending at A ( $dPF_+$ ) and B ( $dPF_-$ ). The fixed point curves are labeled with the signs of their eigenvalues.  $C_-$  is the periodic solution born at the curve  $H_-$ .

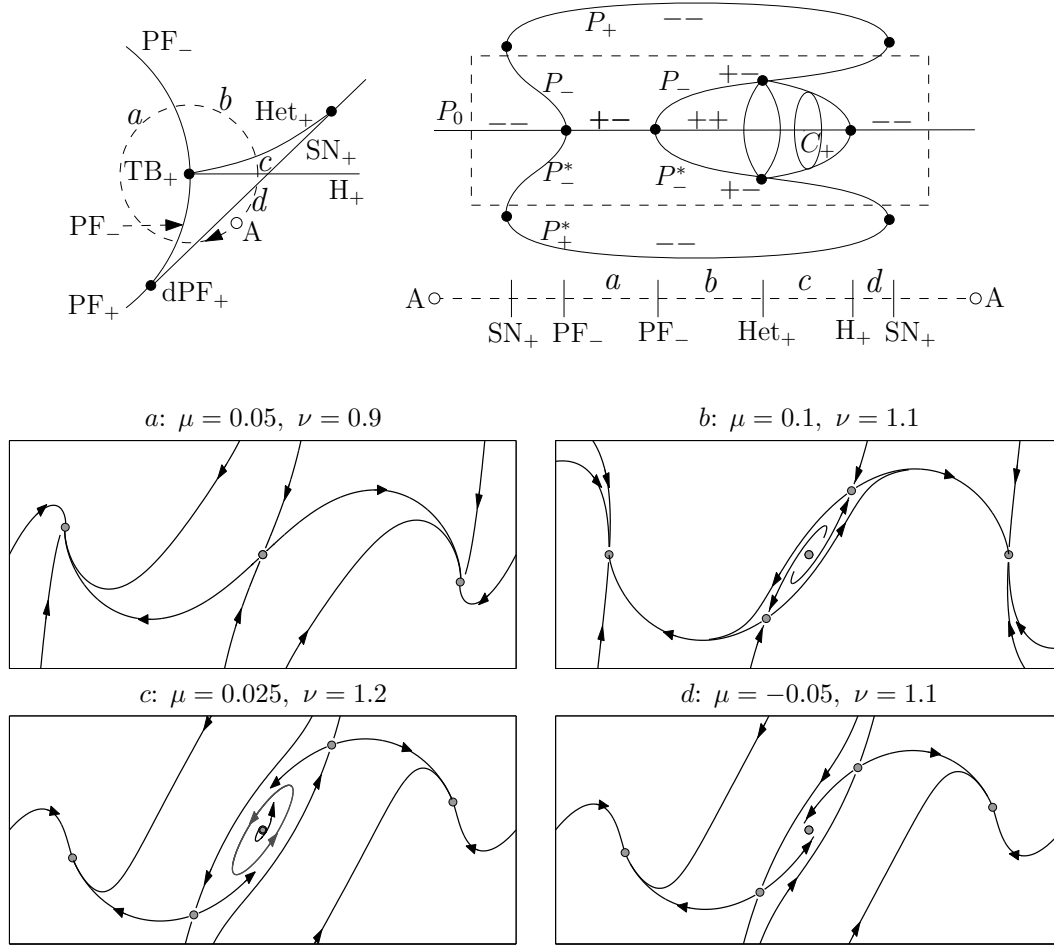


Figure 9: Takens–Bogdanov bifurcation with  $Z_2$  symmetry  $TB_+$ . The top left shows bifurcation curves emanating from  $TB_+$  in parameter space, with a closed one-dimensional path. The top right shows a schematic of the bifurcations along the closed path, starting and ending at  $A$ . The fixed point curves are labeled with the signs of their eigenvalues.  $C_+$  is the periodic solution born at the curve  $H_+$ . The region inside the dashed rectangle on the right contains the states locally connected with the bifurcation  $TB_+$ . The bottom panels show four numerically computed phase portraits, at points labeled  $a$ ,  $b$ ,  $c$  and  $d$ , for the specified parameter values.

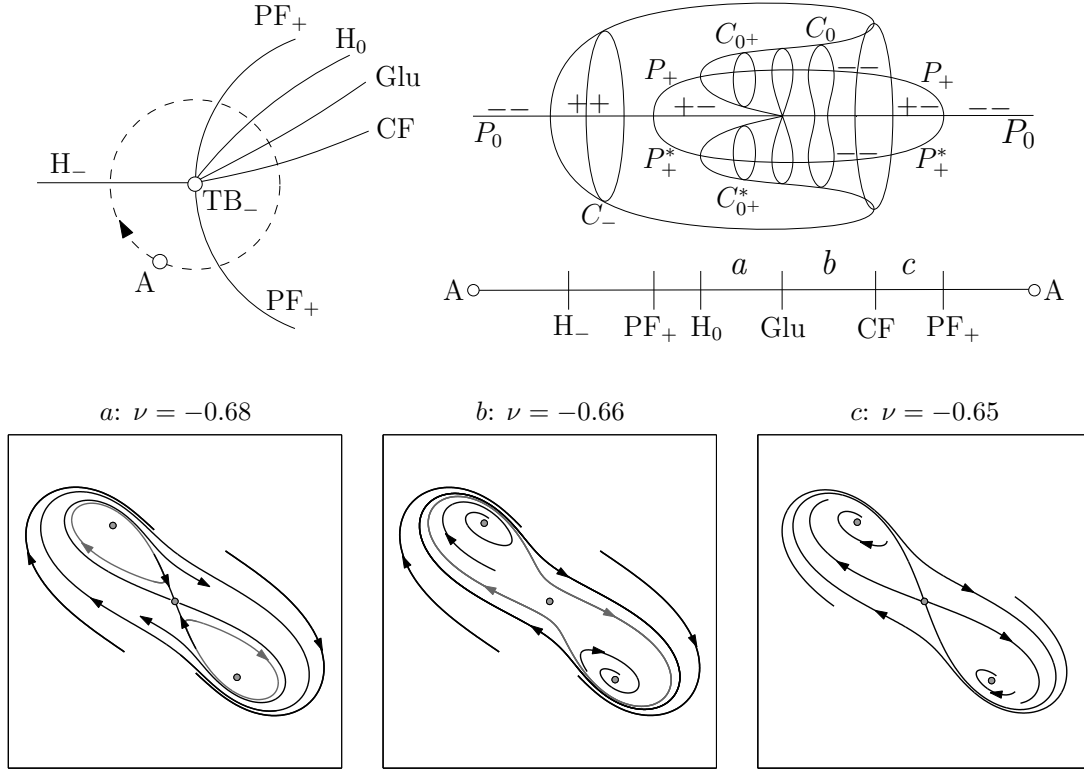


Figure 10: Takens–Bogdanov bifurcation with  $Z_2$  symmetry,  $TB_-$ . The top left shows bifurcation curves emanating from  $TB_-$  in parameter space, with a closed one-dimensional path. The top right shows a schematic of the bifurcations along the closed path, starting and ending at  $A$ . The fixed point curves are labeled with the signs of their eigenvalues.  $C_-$  is the periodic solution born at the curve  $H_-$ ;  $C_{0+}$  and  $C_{0+}^*$  are the unstable cycles born simultaneously at the Hopf bifurcation  $H_0$  around the fixed points  $P_+$  and  $P_+^*$ ; and  $C_0$  is the cycle around both fixed points that remains after the gluing bifurcation. The bottom panels show three numerically computed phase portraits, at points labeled  $a$ ,  $b$  and  $c$ , for  $\mu = 0.5$  and  $\nu$  as indicated, illustrating the gluing and cyclic fold bifurcations.

the  $P_-$  and  $P_-^*$  fixed points along the saddle-node bifurcation curve  $SN_+$ , although they are not locally connected to the codimension-two point, in order to show all the fixed points in the phase space of (19). A curve of global bifurcations, a heteroclinic cycle  $Het_+$  connecting  $P_-$  and  $P_-^*$ , is born at  $TB_+$ . The heteroclinic cycle is formed when the limit cycle  $C_+$  simultaneously collides with the saddles  $P_-$  and  $P_-^*$ .

The scenario  $TB_-$ , is depicted in figure 10, showing the bifurcation diagram and also the bifurcations along a closed one-dimensional path around the codimension-two point. Two curves of global bifurcations are born at  $TB_-$ . One corresponds to a gluing bifurcation  $Glu$ , when the two unstable limit cycles  $C_{0+}$  and  $C_{0+}^*$ , born at the Hopf bifurcation  $H_0$  around  $P_+$  and  $P_+^*$ , simultaneously collide with the saddle  $P_0$ ; after the collision a large cycle  $C_0$  results, surrounding the three fixed points  $P_0$ ,  $P_+$  and  $P_+^*$ . The second global bifurcation curve corresponds to a saddle-node of cycles, where  $C_0$  and  $C_-$  collide and disappear.

A generic Takens–Bogdanov bifurcation (without symmetry) takes place at the  $TB$  point on the  $SN_-$  curve. At the same point in parameter space, but separate in phase space, two  $Z_2$  symmetrically related Takens–Bogdanov bifurcations take place, with  $P_+$  and  $P_+^*$  being the bifurcating states. A schematic of the bifurcations along a one-dimensional path in parameter space around the  $TB$  point is shown in figure 11. Apart from the states locally connected to both  $TB$  bifurcation,

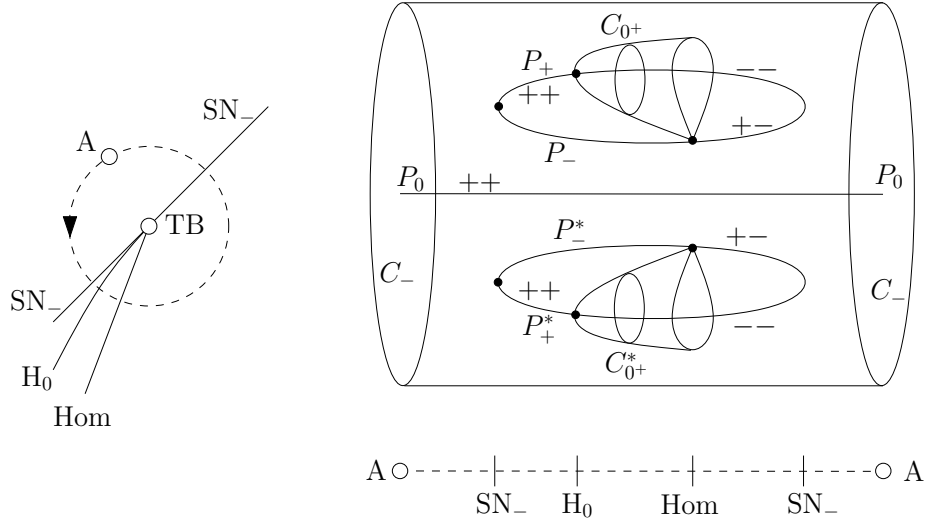


Figure 11: Double Takens–Bogdanov bifurcation TB: left, bifurcation curves emanating from TB in parameter space, with a closed one-dimensional path; right, schematics of the bifurcations along the closed path, starting and ending at A.

there also exist the base state  $P_0$  and the limit cycle  $C_-$ .

### 4.3 Global bifurcations

In the analysis of the local bifurcations of fixed points we have found three curves of global bifurcations, a gluing curve Glu and a saddle-node of cycles CF emerging from  $TB_-$ , a heteroclinic loop born at  $TB_+$  ( $Het_+$ ), and a homoclinic loop emerging from TB (Hom). One wonders about the fate of these global bifurcation curves, and about possible additional global bifurcations. Numerical simulations of the solutions of the normal form ODE system (19), or equivalently (26), together with dynamical systems theory considerations have been used to answer these questions, and a schematic of all local and global bifurcation curves is shown in figure 12.

The gluing bifurcation Glu born at  $TB_-$  and the two homoclinic loops emerging from the two Takens–Bogdanov bifurcations TB (bifurcations of the symmetric fixed points  $P_+$  and  $P_+^*$ ) meet at the point PfGl on the circle  $PF_-$ , where the base state  $P_0$  undergoes a pitchfork bifurcation (see figure 13a). At that point, the two homoclinic loops of the gluing bifurcation, both homoclinic at the same point on the stable  $P_0$  branch, split when the two fixed points  $P_-$  and  $P_-^*$  bifurcate from  $P_0$  (see figure 13b). The two homoclinic loops are then attached to the bifurcated points and separate along the curve Hom. The large unstable limit cycle  $C_0$ , after the pitchfork bifurcation  $PF_-$ , collides simultaneously with both  $P_-$  and  $P_-^*$ , forming a heteroclinic loop along the curve  $Het_0$  (see figure 13c and d2). Both curves Hom and  $Het_0$  are born at PfGl and separate, leaving a region in between where none of the cycles  $C_{0+}$ ,  $C_{0+}^*$  and  $C_0$  exist. The unstable periodic solution  $C_0$  merges with the stable periodic solution  $C_-$  that was born in  $H_-$  and existed in region I, resulting in a cyclic-fold bifurcation of periodic solutions CF. Phase portraits around PfGl are shown in figure 13(d).

The curve Hom born at PfGl ends at the double Takens–Bogdanov point TB. Locally, around both Takens–Bogdanov bifurcations at TB, after crossing the homoclinic curve the limit cycles  $C_{0+}$  and  $C_{0+}^*$  disappear, and no cycles remain. The formation of a large cycle  $C_0$  at  $Het_0$  surrounding both fixed points  $P_+$  and  $P_+^*$  is a global bifurcation involving simultaneously both  $P_-$  and  $P_-^*$  unstable points. It is the re-injection induced by the presence of the  $Z_2$  symmetry that is responsible for this global phenomenon (Adler, 1946, 1973; Strogatz, 1994; Kuznetsov, 2004). The two global bifurcation curves  $Het_0$  and CF become very close when leaving the PfGl neighborhood, and merge



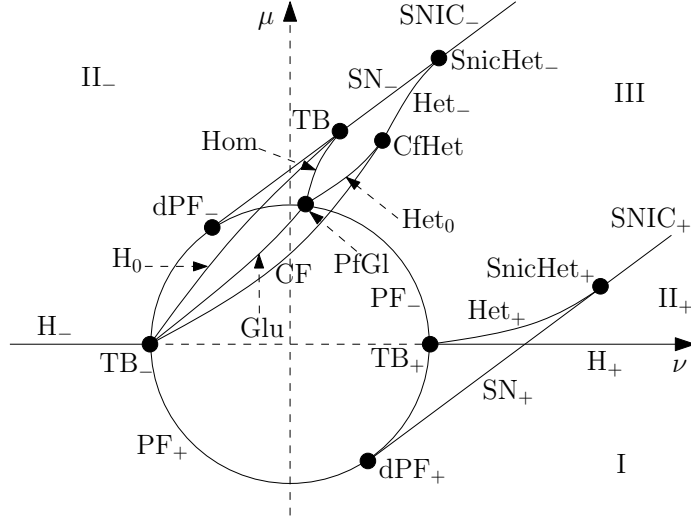


Figure 12: Global bifurcations in the symmetry breaking of  $SO(2)$  to  $Z_2$  case. Codimension-one bifurcation curves: Gluing Glu, cyclic-fold CF, homoclinic collision Hom and heteroclinic loops  $Het_0$ ,  $Het_{\pm}$ . Codimension-two bifurcation points: pitchfork-gluing bifurcation PfGl, cyclic-fold heteroclinic bifurcation CfHet and two SNIC-heteroclinic bifurcations  $SnicHet_{\pm}$ .

at some point in a CfHom (Cyclic-Fold-Heteroclinic collision) codimension-two global bifurcation. After CfHom, the stable limit cycle  $C_-$ , instead of undergoing a cyclic-fold bifurcation, directly collides with the saddle points  $P_-$  and  $P_-^*$  along the  $Het_-$  bifurcation curve (see figure 12). In fact  $Het_0$  and  $Het_-$  are collisions of a limit cycle with  $P_-$  and  $P_-^*$ , but the limit cycle is on a different branch of the saddle-node of cycles CF on each side of CfHom. The two limit cycles born at CF are extremely close together in the neighborhood of CfHom, and it is impossible to see them in a phase portrait, except with a very large zoom around  $P_-$  or  $P_-^*$ .

When increasing  $\mu^2 + \nu^2$ , the heteroclinic loop  $Het_-$  born at CfHet intersects the  $SN_-$  curve at a codimension-two global bifurcation point  $SnicHet_-$ . When they intersect, the saddle-node appears precisely on the limit cycle, resulting in a SNIC bifurcation (a saddle-node on an invariant circle bifurcation). At the  $SnicHet_-$  point, the saddle-node and homoclinic bifurcation curves become tangents, and the saddle-node curve becomes a SNIC bifurcation curve for larger values of  $\mu^2 + \nu^2$ . Figure 14 shows numerically computed phase portraits for  $\alpha_0 = 45^\circ$ , below and above the  $SnicHet_-$ , located at  $\nu \approx 0.1290$ ,  $\mu \approx 1.505$ . In the first case (figure 14a) the saddle-node bifurcation  $SN_-$  takes place in the interior of  $C_-$ , while in the second case (figure 14b)  $SN_-$  happens precisely on top of  $C_-$ , resulting in a SNIC bifurcation.

There remains a global bifurcation curve to be analyzed, the heteroclinic loop  $Het$  born at  $TB_+$ . As shown in figure 12, the curve  $Het$  intersects the  $SN_+$  curve tangentially at a codimension-two global bifurcation point  $SnicHet_+$ . Beyond this point, the  $SN_+$  curve becomes a line of SNIC bifurcations, where the double saddle-node bifurcations appear on the stable limit cycle  $C_+$ , which disappears on entering the pinning region III, exactly in the same way as has been discussed for the  $SnicHet_+$  bifurcation.

We can estimate the width of the pinning region as a function of the magnitude of the imperfection  $\epsilon$  and the distance  $d$  to the bifurcation point,  $w(d, \epsilon)$ . The distance  $d$  will be measured along the line L, and the width  $w(d)$  will be the width of the pinning region measured transversally to L at a distance  $d$  from the origin. It is convenient to use the coordinates  $(u, v)$ , so that the parameter  $u$  along L is precisely the distance  $d$ . If the pinning region is delimited by a curve of equation  $v = \pm h(u)$ , then  $w = 2h(u) = 2h(d)$ . With an imperfection of the form  $\epsilon \bar{z}$ , the case analyzed in this section, the pinning region is of constant width  $w = 2$ . By restoring the dependence on  $\epsilon$ , we obtain a width of value  $w(d, \epsilon) = 2\epsilon$ , independent of the distance to the bifurcation point; the width

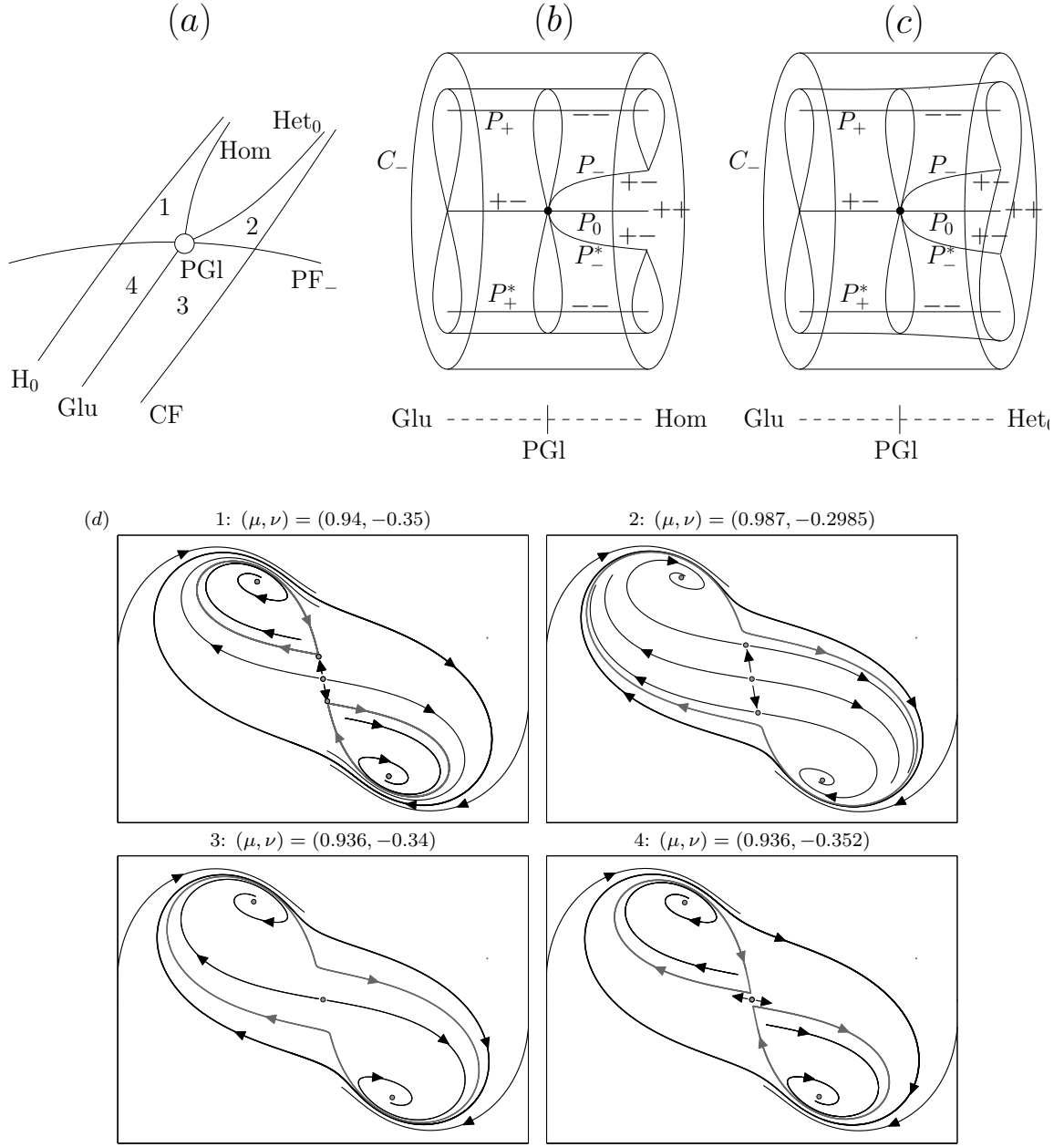


Figure 13: Pitchfork-gluing bifurcation  $\text{PfGl}$ . Top left, bifurcation curves emanating from  $\text{PfGl}$  in parameter space; top right, schematics of the bifurcations along the line  $\text{Glu}-\text{Hom}$ . The bottom row shows four numerically computed phase portraits, at points labeled 1, 2, 3 and 4, for the specified parameter values.

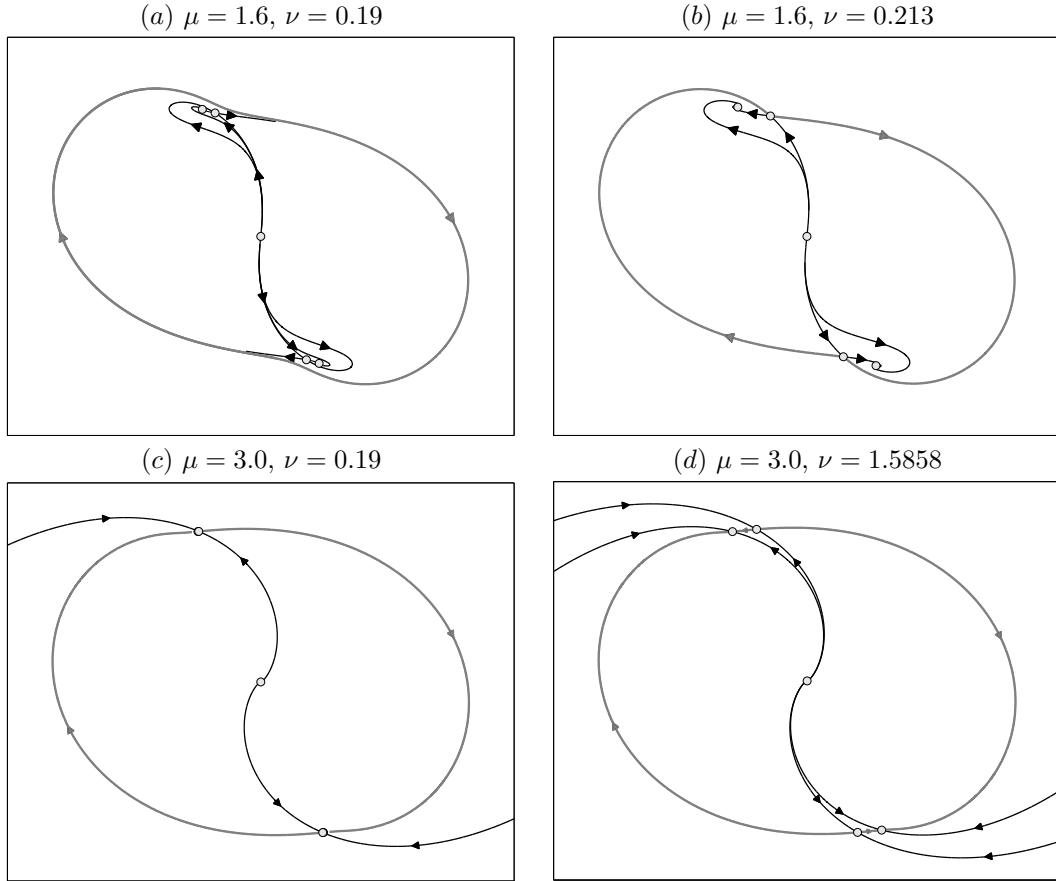


Figure 14: Numerically computed phase portraits in the  $\epsilon\bar{z}$  case, for  $\alpha_0 = 45^\circ$  and  $\mu$  and  $\nu$  as indicated; cases (a) and (b) are below the  $\text{SnicHet}_-$  point and cases (c) and (d) are above the  $\text{SnicHet}_-$  point.

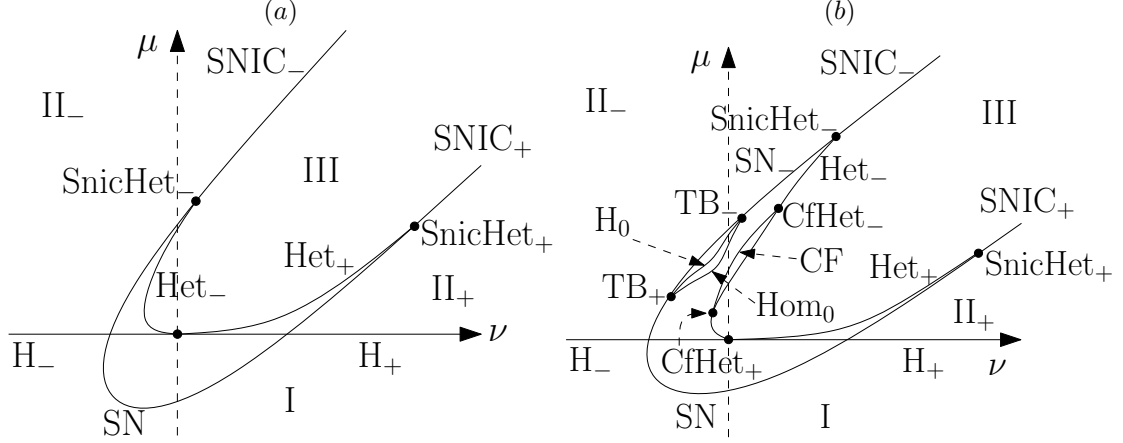


Figure 15: Bifurcation curves corresponding to the normal forms with quadratic terms (29) in the  $\epsilon\bar{z}^2$  case. (a)  $\alpha_0 > 30^\circ$ , (b)  $\alpha_0 < 30^\circ$ .  $H_0$  is tangent to the parabola at the Takens bogdanov points  $TB_\pm$ , and  $H_0$ ,  $\text{Hom}_0$  almost coincide with  $\text{SN}_-$ ; in the figure the distances have been exaggerated for clarity.

of the pinning region is proportional to  $\epsilon$ .

## 5 Symmetry breaking of $SO(2)$ with quadratics terms

### 5.1 The $\epsilon\bar{z}^2$ case

The normal form to be analyzed in this case is

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \bar{z}^2, \quad (29)$$

or in terms of the modulus and phase  $z = re^{i\phi}$ ,

$$\begin{aligned} \dot{r} &= r(\mu - ar^2) + r^2 \cos 3\phi, \\ \dot{\phi} &= \nu - br^2 - r \sin 3\phi. \end{aligned} \quad (30)$$

The fixed points are the origin  $P_0$  ( $r = 0$ ) and the solutions of the same biquadratic equation as in the two previous cases. However, there is an important difference: due to the factor 3 inside the trigonometric functions in (30), the  $P_\pm^i$  points come in triplets ( $i = 1, 2, 3$ ), each triplet has the same radius  $r$  but their phases differ by  $120^\circ$ . This is a consequence of the invariance of the governing equation (29) to the  $Z_3$  symmetry group generated by rotations of  $120^\circ$  around the origin. This invariance was not present in the two previous cases. The bifurcation curves of the fixed points (excluding the Hopf bifurcations of  $P_\pm^i$ ) are still given by figure 17(a), but now a triplet of symmetric saddle-node bifurcations take place simultaneously on the  $\text{SN}_+$  and  $\text{SN}_-$  curves.

It can be seen (details in D) that  $P_-^i$  are saddles in the whole of region III, while  $P_+^i$  are stable, except for small angles  $\alpha_0 < \pi/6$  in a narrow region close to  $\text{SN}_-$  where they are unstable. For  $\alpha_0 > \pi/6$ , the bifurcation diagram is exactly the same as in the  $z\bar{z}$  case (figure 16a), except that the homoclinic curves are now heteroclinic cycles between the triplets of saddles  $P_-^i$ ; this case is illustrated in figure 15(a). For  $\alpha_0 < \pi/6$ , two Takens–Bogdanov bifurcation points appear at the tangency points between the  $\text{SN}_-$  curve and the arc  $H_0$  of the ellipse  $(b\mu - 2a\nu)^2 + (a\mu - 1)^2 = 1$ , as shown in figure 15(b). The arc  $H_0$  is a Hopf bifurcation curve of  $P_+^i$ : three unstable limit cycles  $C_0^i$  are born when the triplet  $P_+^i$  becomes stable. These unstable limit cycles disappear upon colliding with the saddles  $P_-^i$  on a curve of homoclinic collisions,  $\text{Hom}_0$ , that ends at the two Takens–Bogdanov points  $TB_+$  and  $TB_-$ . This situation is very similar to what happens in

the  $\epsilon\bar{z}$  case analyzed in section §4, where a Hopf curve  $H_0$  appeared close to  $SN_-$  joining two Takens–Bogdanov points. In both cases, the  $SO(2)$  symmetry is not completely broken, but a  $Z_m$  symmetry remains. For  $\alpha_0 = \pi/6$ , the ellipse  $H_0$  becomes tangent to  $SN_-$  and the two Takens–Bogdanov points coalesce, disappearing for  $\alpha_0 > \pi/6$ .

Finally, the stable limit cycles  $C_-$  and  $C_+$ , born at the Hopf bifurcations  $H_-$  and  $H_+$ , upon entering region III collide simultaneously with the saddles  $P_+^i$ ,  $i = 1, 2$  and  $3$ , and disappear along two heteroclinic bifurcation curves  $Het$  and  $Het'$  for small values of  $\mu$ . These curves collide with the parabola at the codimension-two bifurcation points  $SnicHet$  and  $SnicHet'$ . For larger values of  $\mu$  the limit cycles  $C_-$  and  $C_+$  undergo SNIC bifurcations on the parabola. The two curves  $Hom_\pm$  emerge from the origin, as in the previous cases.

In the three quadratic cases, the pinning region is delimited by the same parabola  $u = v^2 - 1/4$ , using the  $(u, v)$  coordinates introduced in (13) (see also figure 2(a)). The width of the pinning region is easy to compute, and is given by  $w = 2v = 2\sqrt{u + 1/4} \sim 2\sqrt{d}$ . By using (11), the dependence on  $\epsilon$  is restored, resulting in  $w(d, \epsilon) = 2\epsilon\sqrt{d}$ . The width of the pinning region increases with the distance  $d$  to the bifurcation point, and it is proportional to the amplitude of the imperfection  $\epsilon$ .

## 5.2 The $\epsilon z\bar{z}$ case

The normal form to be analyzed in this case is

$$\dot{z} = z(\mu + i\nu - c|z|^2) + z\bar{z}, \quad (31)$$

or in terms of the modulus and phase  $z = re^{i\phi}$ ,

$$\begin{aligned} \dot{r} &= r(\mu - ar^2) + r^2 \cos \phi, \\ \dot{\phi} &= \nu - br^2 - r \sin \phi. \end{aligned} \quad (32)$$

There are three fixed points: the origin  $P_0$  ( $r = 0$ ) and the two solutions  $P_\pm$  of the biquadratic equation  $r^4 - 2(a\mu + b\nu + 1/2)r^2 + \mu^2 + \nu^2 = 0$ , given by

$$r_\pm^2 = a\mu + b\nu + \frac{1}{2} \pm \sqrt{a\mu + b\nu + \frac{1}{4} - (a\nu - b\mu)^2}. \quad (33)$$

These solutions are born at the parabola  $a\mu + b\nu + 1/4 = (a\nu - b\mu)^2$ , and exist only in its interior, which is the pinning region III in figure 16. The parabola is a curve of saddle-node bifurcations. It can be seen (details in D) that  $P_+$  is stable while  $P_-$  is a saddle in the whole of region III, so there are no additional bifurcations of fixed points in the  $z\bar{z}$  case, in contrast to the  $z^2$  case.

As the perturbation is of second order, the Jacobian at  $P_0$  is the same as in the unperturbed case, and the Hopf bifurcations of  $P_0$  take place along the horizontal axis  $\mu = 0$ . As in the unperturbed case, the Hopf frequency is negative for  $\nu < 0$  ( $H_-$ ), it is zero at the origin, and becomes positive for  $\nu > 0$  ( $H_+$ ). The Hopf curves  $H_-$  and  $H_+$  extend in this case up to the origin, in contrast with the zero and first order cases examined previously, where the Hopf curves ended in Takens–Bogdanov bifurcations without reaching the origin.

The stable limit cycles  $C_-$  and  $C_+$ , born at the Hopf bifurcations  $H_-$  and  $H_+$ , upon entering region III collide with the saddle  $P_+$  and disappear along two homoclinic bifurcation curves  $Hom_\pm$  for small values of  $\mu$ . For larger values of  $\mu$ , the curves  $Hom_\pm$  collide with the parabola at the codimension-two bifurcation points  $SnicHom_\pm$ , and for larger values of  $\mu$ , the saddle-node bifurcations take place on the parabola and the limit cycles  $C_-$  and  $C_+$  undergo SNIC bifurcations, as in the previous  $\epsilon$  and  $\epsilon\bar{z}$  cases. Figure 16(a) summarizes all the local and global bifurcation curves in the  $z\bar{z}$  case, and shows numerically computed phase portraits around the  $SnicHom_-$  point. Figure 16(b) shows the  $SN_+$  and  $Hom_-$  bifurcations before  $SnicHom_-$  (at  $\mu = 0.03$ ), and figure 16(c) illustrates the  $SNIC_-$  bifurcation after  $SnicHom_-$  (at  $\mu = 0.033$ ).

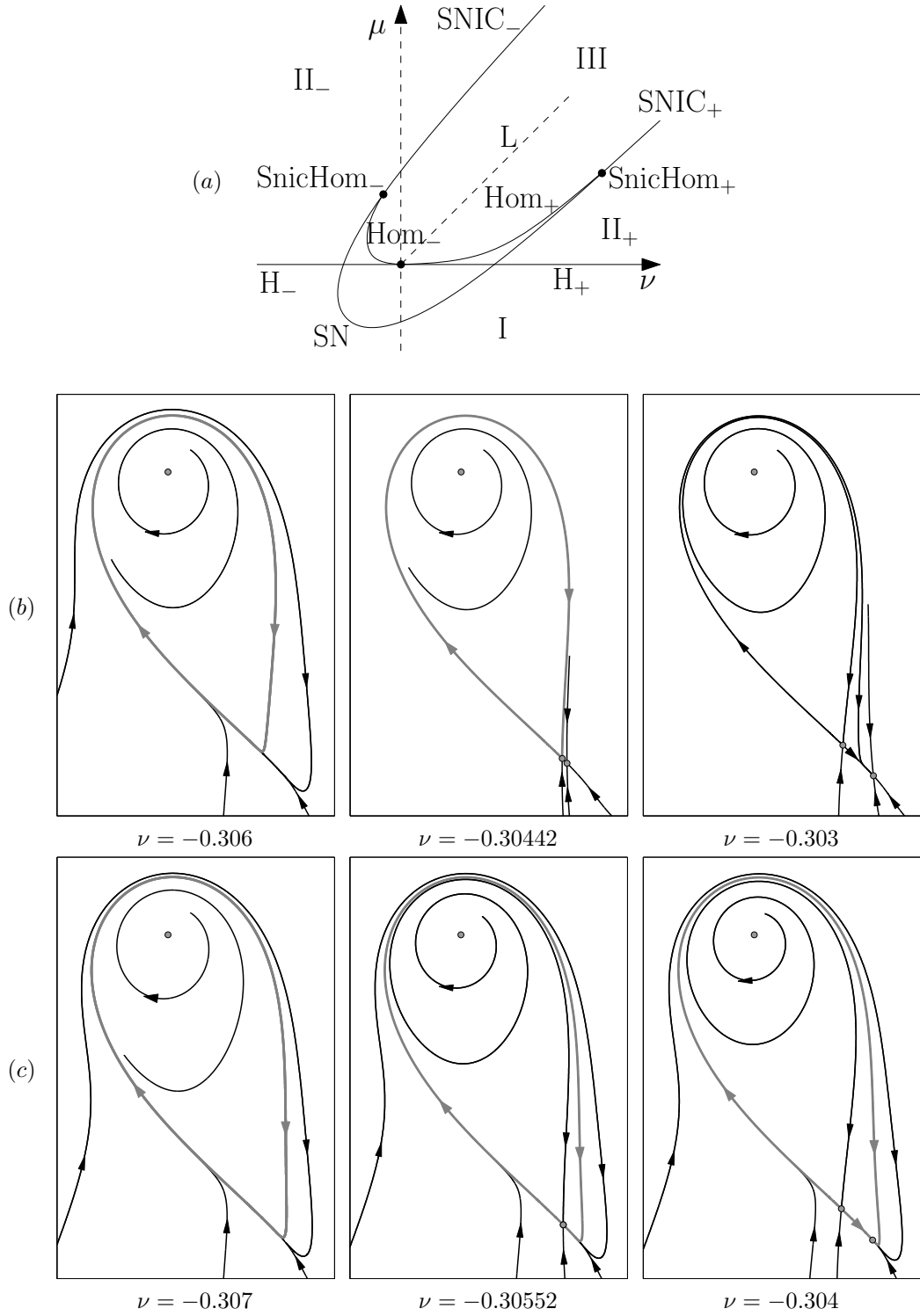


Figure 16: (a) Schematic of bifurcation curves corresponding to the normal form with quadratic terms in the  $\epsilon z \bar{z}$  case. Phase portraits (b) crossing the  $Hom_-$  curve at  $\mu = 0.03$ , for  $\nu$  values as specified, and (c) crossing the  $SNIC_-$  curve at  $\mu = 0.033$ . Thick grey lines correspond to the periodic orbit and the homoclinic and heteroclinic loops.

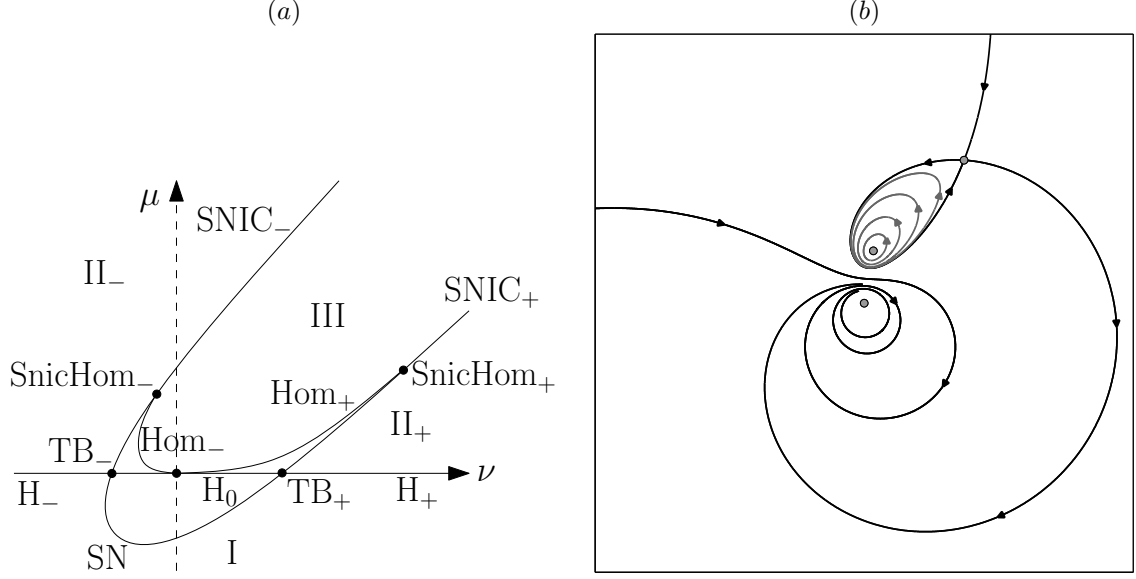


Figure 17: (a) Bifurcation curves corresponding to the normal form with quadratic terms in the  $\epsilon z^2$  case. (b) Phase portrait on the curve  $H_0$  showing the degenerate Hopf bifurcation and the associated homoclinic loop.

### 5.3 The $\epsilon z^2$ case

The normal form to be analyzed in this case is (10) with  $p = q = 2$  and  $\epsilon = 1$ :

$$\dot{z} = z(\mu + i\nu - c|z|^2) + z^2. \quad (34)$$

The normal form (34), in terms of the modulus and phase  $z = re^{i\phi}$ , reads

$$\begin{aligned} \dot{r} &= r(\mu - ar^2) + r^2 \cos \phi, \\ \dot{\phi} &= \nu - br^2 + r \sin \phi. \end{aligned} \quad (35)$$

There are three fixed points: the origin  $P_0$  ( $r = 0$ ) and the two solutions  $P_{\pm}$  of the biquadratic equation  $r^4 - 2(a\mu + b\nu + 1/2)r^2 + \mu^2 + \nu^2 = 0$ , which are the same as in the  $\epsilon z\bar{z}$  case (§5.2). In fact, the fixed points in these two cases have the same modulus  $r$  and their phases have opposite sign; changing  $\phi \rightarrow -\phi$  in (32) results in (35). Therefore, the bifurcation curves of the fixed points (excluding Hopf bifurcations of  $P_{\pm}$ ) in this case are also given by figure 16(a).

It can be seen (details in D) that  $P_+$  is a saddle in the whole of region III, while  $P_-$  is stable for  $\mu > 0$ , unstable for  $\mu < 0$ , and undergoes a Hopf bifurcation  $H_0$  along the segment of  $\mu = 0$  delimited by the parabola of saddle-node bifurcations. The points  $TB_{\pm}$  where  $H_0$  meets the parabola are Takens–Bogdanov codimension-two bifurcations. Figure 17 summarize all the local and global bifurcation curves just described.

In the present case, the two Takens–Bogdanov bifurcations and the Hopf bifurcations along  $H_0$  are degenerate, as shown in D. Detailed analysis and numerical simulations show that the Hopf and homoclinic bifurcation curves emerging from the Takens–Bogdanov point are both coincident with the  $H_0$  curve previously mentioned. Moreover, the interior of the homoclinic loop is filled with periodic orbits, and no limit cycle exists on either side of  $H_0$ . This situation is illustrated in the phase portrait in figure 17(b). This highly degenerate situation will be broken by the presence of additional terms in the normal form, and of the continuous family of periodic orbits, only a few will remain. Dumortier *et al.* (1987), who have analyzed in detail the unfolding of such a degenerate case, find that at most two of the periodic orbits survive.

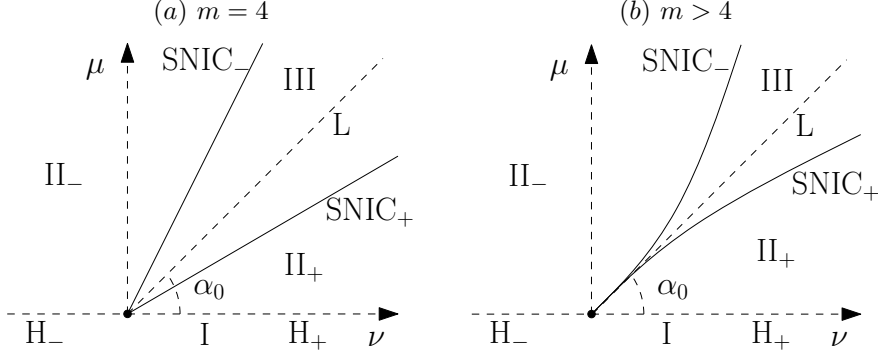


Figure 18: Bifurcation curves corresponding to the normal forms retaining a  $Z_m$  symmetry (37); (a) corresponds to the  $m = 4$  case and (b) corresponds to the  $m > 4$  cases.

Finally, and exactly in the same way as in the  $z\bar{z}$  case examined in the previous subsection, the stable limit cycles  $C_-$  and  $C_+$  born respectively at the Hopf bifurcations  $H_-$  and  $H_+$  and existing in regions  $I_-$  and  $I_+$  (figure 17a), on entering region III collide with the saddle  $P_+$  and disappear along two homoclinic bifurcation curves  $\text{Hom}_\pm$  for small values of  $\mu$ . These curves collide with the parabola at the codimension-two bifurcation points  $\text{SnHom}_\pm$  and for larger values of  $\mu$ , the limit cycles  $C_-$  and  $C_+$  undergo SNIC bifurcations on the parabola. The two curves  $\text{Hom}_\pm$  emerge from the origin, as in the previous case.

## 6 Symmetry breaking $SO(2) \rightarrow Z_m$ , $m \geq 4$

For completeness, and also for intrinsic interest, we will explore the breaking of the  $SO(2)$  symmetry to  $Z_m$ , so that the imperfections added to the normal form (6) preserve the  $Z_m$  subgroup of  $SO(2)$  generated by rotations of  $2\pi/m$  (also called  $C_m$ ). The lowest order monomial in  $(z, \bar{z})$  not of the form  $z|z|^{2p}$  and equivariant under  $Z_m$ , is  $\bar{z}^{m-1}$ , resulting in the normal form

$$\dot{z} = z(\mu + i\nu - c|z|^2) + \epsilon\bar{z}^{m-1}. \quad (36)$$

In terms of the modulus and phase of the complex amplitude  $z = re^{i\phi}$ , the normal form becomes

$$\begin{aligned} \dot{r} &= r(\mu - ar^2) + \epsilon r^{m-1} \cos m\phi, \\ \dot{\phi} &= \nu - br^2 - \epsilon r^{m-2} \sin m\phi. \end{aligned} \quad (37)$$

The cases  $m = 2$  and  $m = 3$  have already been examined in §4 and §5.1 respectively. When  $m \geq 4$ , the term  $\epsilon\bar{z}^{m-1}$  is smaller than the remaining terms in (36), so the effect of the symmetry breaking is going to be small compared with the other cases analyzed in this paper. The fixed point solutions of (37), apart from the trivial solution  $P_0$  ( $r = 0$ ), are very close the zero-frequency line L in the perfect system. Using the coordinates  $(u, v)$  along and orthogonal to L (13), the nontrivial fixed points of (37) satisfy

$$(r^2 - u)^2 = \epsilon^2 r^{2m-4} - v^2. \quad (38)$$

On L,  $r^2 = u$ ; close to L, the fixed points  $P_\pm$  are given by  $r^2 \sim u \pm \sqrt{\epsilon^2 u^{m-2} - v^2}$ . The pinning region, at dominant order in  $\epsilon$ , is  $v = \pm \epsilon u^{(m-2)/2}$ . This gives a wedge-shaped region around L for  $m = 4$  and a horn for  $m > 4$ , as illustrated in figure 18. On the boundaries of the pinning region, the fixed points merge in saddle-node bifurcations that take place on the limit cycles  $C_\pm$ . These are curves of SNIC bifurcations, exactly the same phenomena that is observed in Neimark-Sacker bifurcations (Arrowsmith & Place, 1990), and that we have encountered also in the previous cases analyzed in the present study. Due to the symmetry  $Z_m$ , from (37) we see that at the boundaries of the wedge,  $m$  simultaneous saddle-node bifurcations take place. Figure 19 shows how the fixed



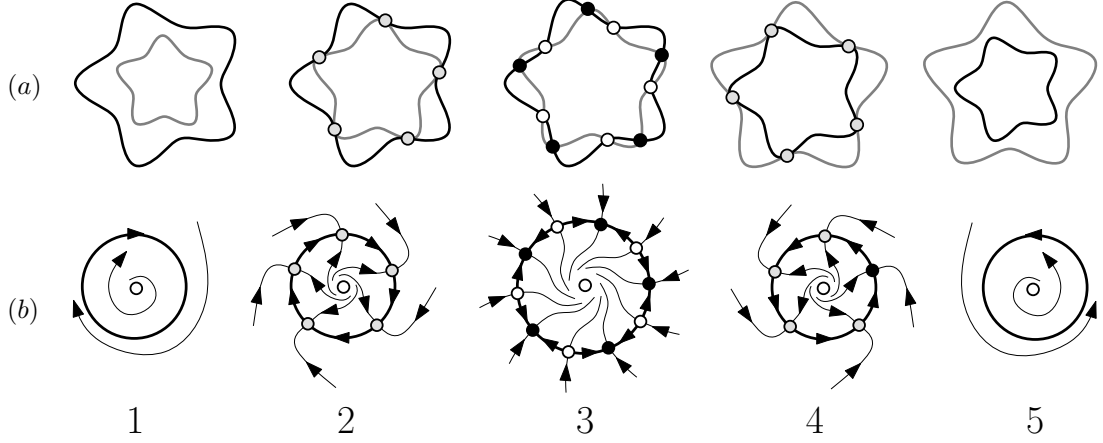


Figure 19: Crossing the horn for  $m = 5$ . (a) Fixed point solutions of (37) at the five points in figure 18(b); grey points in 2 and 4 are the saddle-node points, that split in a stable point (black) and saddle (white). (b) Phase portraits corresponding to the five cases in (a).

points appear and disappear in saddle-node bifurcations on the limit cycle  $C_{\pm}$  when crossing the horn for the  $m = 5$  case, at the five points in parameter space indicated in figure 18(b). The nontrivial fixed points, from (37) and  $r^2 \sim \mu/a \sim \nu/b$ , satisfy

$$\begin{aligned} r^2 &= \mu/a + \epsilon(\mu/a)^{(m-1)/2} \cos m\phi, \\ r^2 &= \nu/b - \epsilon(\nu/b)^{(m-1)/2} \sin m\phi, \end{aligned} \quad (39)$$

and the intersection of these circles modulated by the  $\sin m\phi$  and  $\cos m\phi$  terms is illustrated in figure 19(a). Phase portraits corresponding to the five points in parameter space are also schematically shown in figure 19(b).

The width of the pinning region in the symmetry breaking  $SO(2) \rightarrow Z_m$  case is obtained from the shape of the horn region and is given by  $w(d, \epsilon) = 2v = 2\epsilon d^{(m-2)/2}$ . Again, as in all the preceding cases, the width of the pinning region is proportional to the amplitude of the imperfection  $\epsilon$ .

## 7 Common features in the different ways to break $SO(2)$ symmetry

Here we summarize the features that are common to the different perturbations analyzed in the previous sections. The most important feature is that the curve of zero frequency splits into two curves with a region of zero-frequency solutions appearing in between (the so-called pinning region). Of the infinite number of steady solutions that exist along the zero-frequency curve in the *perfect* system with  $SO(2)$  symmetry, only a small finite number remain. These steady solutions correspond to the pinned solutions observed in experiments and in numerical simulations, like the ones to be described in §8. The number of remaining steady solutions depends on the details of the symmetry-breaking imperfections, but when  $SO(2)$  is completely broken and no discrete symmetries remain, there are three steady solutions in the pinning region III (see figure 20a). One corresponds to the base state, now unstable with eigenvalues  $(+, +)$ . The other two are born on the SNIC curves delimiting region III away from the origin. Of these two solutions, one is stable (the only observable state in region III) and the other is a saddle (see figure 20b and c). There are also the two Hopf bifurcation curves  $H_-$  and  $H_+$ . The regions where the Hopf bifurcations meet the infinite-period bifurcations cannot be described in general, and as has been shown in the examples in the previous sections, will depend on the specifics of how the  $SO(2)$  symmetry is broken, i.e. on the specifics of the imperfections present in the problem considered. These regions contain complex bifurcational

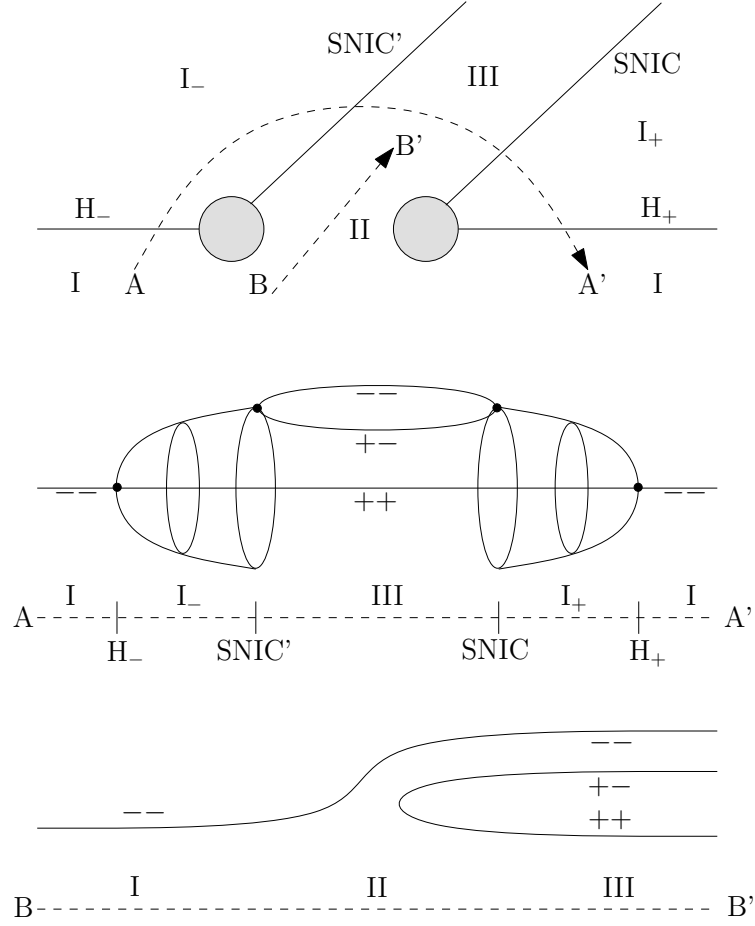


Figure 20: Imperfect Hopf under general perturbations: (a) regions in parameter space; (b) and (c), bifurcation diagrams along the two one-dimensional paths, (1) and (2), respectively. The signs ( $++$ ,  $--\dots$ ) indicate the sign of the real part of the two eigenvalues of the solution branch considered.

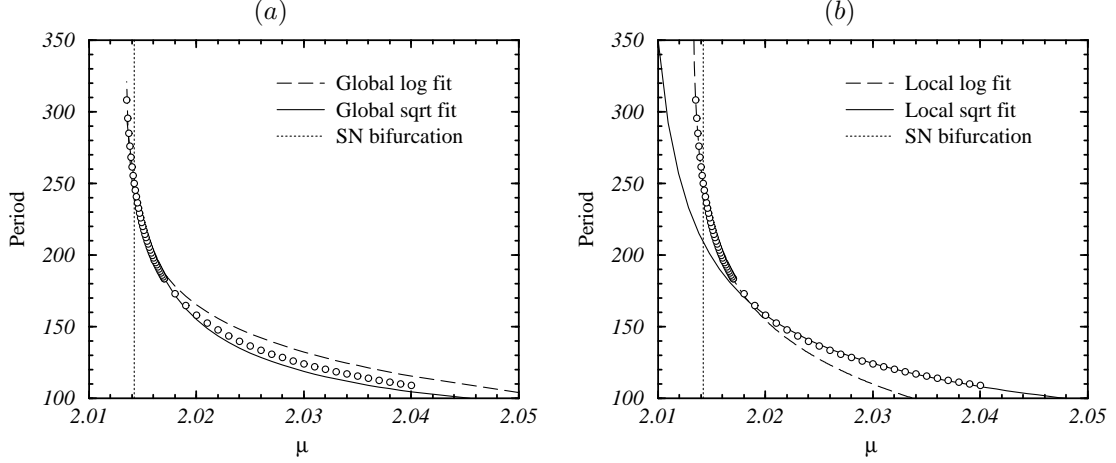


Figure 21: Square root and logarithmic fits to the periods of  $C_-$  approaching the  $\text{Hom}_-$  curve at  $\nu = 0.6$ : (a) fitted curves in the complete range  $\mu \in [2.0135, 2.2]$ , and (b) fits in the range  $\mu \in [2.0135, 2.027]$  for the log fit and  $\mu \in [2.018, 2.2]$  for the square root fit.

processes, and are represented as grey disks in figure 20(a). The stable limit cycle existing outside III, in regions  $I_{\pm}$ , undergoes a SNIC bifurcation and disappears upon entering region III (see figure 20b). When the SNIC bifurcation curves approach the Hopf bifurcation curves (i.e. enter the grey disks regions), the saddle-node bifurcations do not occur on the stable limit cycle but very close, and the limit cycle disappears in a saddle-loop homoclinic collision that occurs very close to the saddle-node bifurcations. These homoclinic collisions behave like a SNIC bifurcation, except in a very narrow region in parameter space around the saddle-node curves, as has been discussed in §4.3.

In all cases considered, the width of the pinning region scales linearly with the strength of the symmetry breaking  $\epsilon$ . In all cases, we have found  $w(d, \epsilon) = 2\epsilon d^{(p-1)/2}$ , where  $p$  is the order of the symmetry breaking considered. For lower order terms, the width decreases ( $\epsilon$  case, order zero) or remains constant ( $\epsilon\bar{z}$  case, order one) with increasing distance from the bifurcation point. For quadratic and higher order terms, the width increases with the distance. When arbitrary perturbations are included, we expect a behavior of the form  $w(d, \epsilon) = \epsilon f(d)$ , where the function  $f$  will depend on the details of the symmetry-breaking terms involved. The size of the regions containing complex bifurcational processes (the grey disks in figure 20a) is of order  $\epsilon$  or smaller, as we have seen in all cases considered. Therefore these regions are comparable in size or smaller than the width of the pinning region.

### 7.1 SNIC versus homoclinic collision: scaling of the period

Another feature we have found in the five scenarios discussed above is that the  $\text{SNIC}_{\pm}$  bifurcations, where the stable limit cycle disappears on entering the pinning region away from the origin ( $\mu = \nu = 0$ ), become homoclinic (or heteroclinic if the  $SO(2)$  symmetry is not completely broken) collisions between the stable limit cycle and the saddle point that is born in the saddle-node bifurcations  $\text{SN}_{\pm}$  that now take place not on the limit cycle but very close to it. There is a codimension-two global bifurcation, termed  $\text{SnicHom}$  in the bifurcation diagrams discussed above, where the SNIC curve, the saddle-node curve and the homoclinic collision curve meet.

The scaling laws of the periods when approaching a homoclinic or a SNIC bifurcation are different, having logarithmic or square root profiles:

$$T_{\text{Het}} = \frac{1}{\lambda} \ln \frac{1}{\mu - \mu_c} + O(1), \quad T_{\text{SNIC}} = \frac{k}{\sqrt{\mu - \mu_c}} + O(1), \quad (40)$$

where  $\lambda$  is the positive eigenvalue of the saddle, and  $k$  a constant. So the question is what happens with the scaling of the period close to the SnicHom bifurcation?

We have analyzed in detail the  $\epsilon\bar{z}$  case, being the other cases very similar. We refer to the figures 12 and 14. From a practical point of view, close to but before the SnicHet<sub>-</sub> point, the saddle-node bifurcation SN<sub>-</sub> is very closely followed by the heteroclinic collision of the limit cycle  $C_-$  with the saddles  $P_-$  and  $P_-^*$ , and it becomes almost indistinguishable from the SNIC<sub>-</sub> bifurcation. We have numerically computed the period of  $C_-$  at  $\nu = 0.6$  for decreasing  $\mu$  values approaching SN<sub>-</sub> in the range  $\mu \in [2.0135, 2.2]$ , for  $\alpha_0 = 45^\circ$ . Figure 21(a) shows both fits using the values of the period over the whole computed range. The log fit overestimates the period while the square-root fit underestimates it, and this underestimate gets larger as the heteroclinic collision is approached. Figure 21(b) again shows both fits, but now using values close to the collision for the log fit and values far away from the collision for the square-root fit. Both fits are now very good approximations of the period in their corresponding intervals, and together cover all the values numerically computed. When the interval between the SN<sub>-</sub> bifurcation and the heteroclinic collision (in figure 21,  $\mu_{\text{SN}} = 2.01420$  and  $\mu_c = 2.01336$ , respectively) is very small, it cannot be resolved experimentally (or even numerically in an extended systems with millions of degrees of freedom, as is the case in fluid dynamics governed by the three-dimensional Navier–Stokes equations). In such a situation the square-root fit looks good enough, because away from the SN<sub>-</sub> point, the dynamical system just feels the ghost of the about to be formed saddle-node pair and does not distinguish between whether the saddle-node appears on the limit cycle or very close to it. However, if we are able to resolve the very narrow parameter range between the saddle-node formation and the subsequent collision with the saddle, then the log fit matches the period in this narrow interval much better.

Due to the presence of two very close bifurcations (Het' and SN<sub>-</sub>), the scaling laws become cross-contaminated, and from a practical point of view the only way to distinguish between a SNIC and a Homoclinic collision is by computing or measuring periods extremely close to the infinite-period bifurcation point. We can also see this from the log fit equation in (40); when both bifurcations are very close,  $\lambda$ , the positive eigenvalue of the saddle, goes to zero (it is exactly zero at the saddle-node point), so the log fit becomes useless, except when the periods are very large.

When the interval between the SN<sub>-</sub> bifurcation and the homoclinic collision is very small, it cannot be resolved experimentally (or even numerically in an extended systems with millions of degrees of freedom, as is the case in fluid dynamics governed by the three-dimensional Navier–Stokes equations). In such a situation, the square-root fit appears to be good enough, because away from the SN<sub>-</sub> point, the dynamical system just feels the ghost of the about-to-be-formed saddle-node pair and does not distinguish between whether the saddle-node appears on the limit cycle or just very close to it. However, if we are able to resolve the very narrow parameter range between the saddle-node formation and the subsequent collision with the saddle, then the log fit matches the period in this narrow interval much better.

## 7.2 Codimension-two bifurcations of limit cycles

The bifurcations that a limit cycle can undergo have been an active subject of research since dynamical systems theory was born. Even in the case of isolated codimension-one bifurcations, a complete classification was not completed until fifteen years ago (Turaev & Shilnikov, 1995), when the blue-sky catastrophe was found. The seven possible bifurcations are: the Hopf bifurcation, where a limit cycle shrinks to a fixed point, and the length of the limit cycle reduces to zero. Three bifurcations where both the length and period of the limit cycle remain finite: the saddle node of cycles (or cyclic fold), the period doubling and the Neimark-Sacker bifurcations. Two bifurcations where the length remains finite but the period goes to infinity: the collision of the limit cycle with an external saddle forming a homoclinic loop, and the appearance of a saddle-node of fixed points on the limit cycle (the SNIC bifurcation). Finally, there is the blue-sky bifurcation where both the length and period go to infinity, corresponding to the appearance of a saddle-node of limit cycles transverse to the given limit cycle. The seven bifurcations are described in many books on dynamical systems, e.g. Shil'nikov *et al.* (2001); Kuznetsov (2004); they are also described on

the web page [http://www.scholarpedia.org/article/Blue-sky\\_catastrophe](http://www.scholarpedia.org/article/Blue-sky_catastrophe) maintained by A. Shil'nikov and D. Turaev.

Of the seven bifurcations, only four (Hopf, cyclic fold, homoclinic collision and SNIC) are possible in planar systems, as is the case in the present study, and we have found the four of them in the different scenarios explored. We have also found a number of codimension-two bifurcations of limit cycles. For these bifurcations a complete classification is still lacking, and it is interesting to list them because some of the bifurcations obtained are not very common. The codimension-two bifurcations of limit cycles associated to codimension-two bifurcations of fixed points can be found in many dynamical systems textbooks, and include Takens–Bogdanov bifurcations (present in almost all cases considered here) and the Bautin bifurcation (in the  $\epsilon$  case). Codimension-two bifurcations of limit cycles associated only to global bifurcations are not so common. We have obtained five of them, that we briefly summarize here.

**PfGl** A gluing bifurcation with the saddle point undergoing a pitchfork bifurcation; it may happen in systems with  $Z_2$  symmetry; see §4.

**CfHom** A cyclic-fold and a homoclinic collision occurring simultaneously; see §4.

**CfHet** A cyclic-fold and a heteroclinic collision occurring simultaneously; see §3.

**SnicHom** A SNIC bifurcation and a homoclinic collision occurring simultaneously; see §5.

**SnicHet** A double SnicHom bifurcation occurring simultaneously in  $Z_2$  symmetric systems; see §4.

The SnicHom bifurcation is particularly important in our problem because it separates the two possible scenarios upon entering the pinning region: the stable limit cycle outside may disappear in a homoclinic collision or a SNIC bifurcation.

## 8 Fluid dynamics examples of pinning due to breaking the $SO(2)$ symmetry

### 8.1 Pinning in small aspect ratio Taylor–Couette flow

Experiments in small aspect-ratio Taylor–Couette flows have reported the presence of a band in parameter space where rotating waves become steady non-axisymmetric solutions (a pinning effect) via infinite-period bifurcations (Pfister *et al.*, 1991). Previous numerical simulations, assuming  $SO(2)$  symmetry of the apparatus, were unable to reproduce these observations (Marques & Lopez, 2006). Recent additional experiments suggest that the pinning effect is not intrinsic to the dynamics of the problem, but rather is an extrinsic response induced by the presence of imperfections that break the  $SO(2)$  symmetry of the ideal problem. Additional controlled symmetry-breaking perturbations were introduced into the experiment by tilting one of the endwalls (Abshagen *et al.*, 2008). Pacheco *et al.* (2011) conducted direct numerical simulations of the Navier–Stokes equations including the tilt of one endwall by a very small angle. Those simulations agree very well with the experiments, and the normal form theory developed in this paper provides a theoretical framework for understanding the observations. A brief summary of those results follows.

Taylor–Couette flow consists of a fluid confined in an annular region with inner radius  $r_i$  and outer radius  $r_o$ , capped by endwalls a distance  $h$  apart. The endwalls and the outer cylinder are stationary, and the flow is driven by the rotation of the inner cylinder at constant angular speed  $\Omega$  (see figure 22 for a schematic). The system is governed by three parameters:

$$\begin{aligned} \text{the Reynolds number} \quad Re &= \Omega r_i (r_o - r_i) / \nu, \\ \text{the aspect ratio} \quad \Gamma &= h / (r_o - r_i), \\ \text{the radius ratio} \quad \eta &= r_i / r_o, \end{aligned} \tag{41}$$

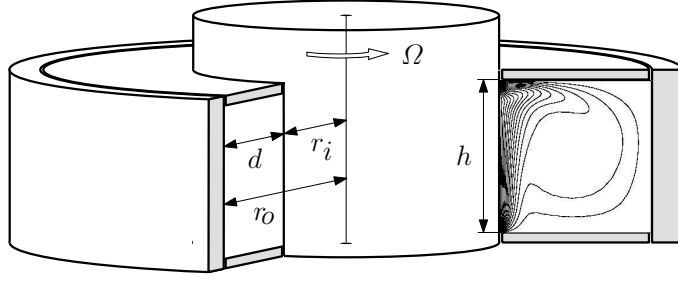


Figure 22: Schematic of the Taylor-Couette apparatus.

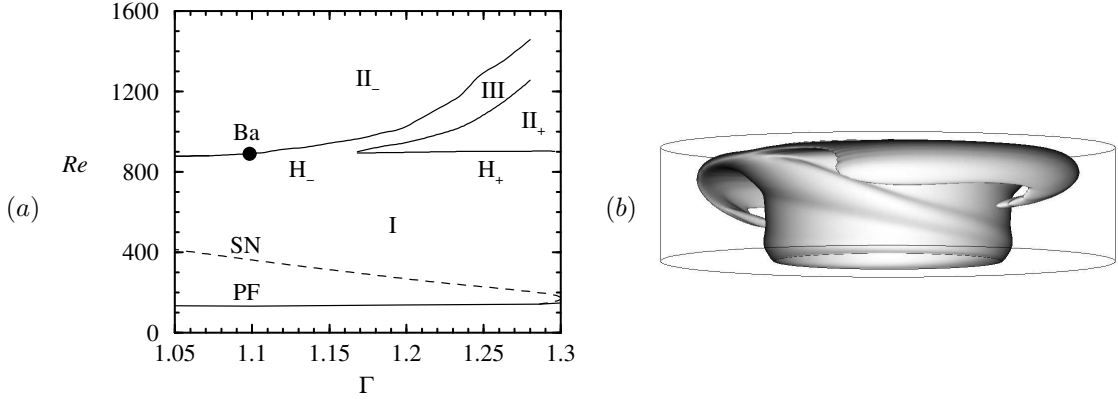


Figure 23: (a) Experimental regimes found in the small aspect-ratio Taylor-Couette problem, with a pinning region, adapted from Pfister *et al.* (1991). (b) Numerically computed rotating wave from Marques & Lopez (2006), found in regions  $\text{II}_{\pm}$ .

where  $\nu$  is the kinematic viscosity of the fluid. The system is invariant to arbitrary rotations about the axis,  $SO(2)$  symmetry, and to reflections about the mid-height, a  $Z_2$  symmetry that commutes with  $SO(2)$ . In both the experiments and the numerical simulations, the radius ratio was kept fixed at  $\eta = 0.5$ .  $Re$  and  $\Gamma$  were varied, and these correspond to the parameters  $\mu$  and  $\nu$  in the normal forms studied here.

For small  $Re$ , below the curve PF in figure 23(a), the flow is steady, axisymmetric and reflection symmetric, consisting of two Taylor vortices (Marques & Lopez, 2006). The  $Z_2$  reflection symmetry is broken in a pitchfork bifurcation along the curve PF, and a pair of steady axisymmetric one-vortex states that have a jet of angular momentum emerging from the inner cylinder boundary layer near one or other of the endwalls is born. Both are stable, and which is realized depends on initial conditions. The only symmetry of these symmetrically-related solutions is  $SO(2)$ . The inset in figure 22 shows the azimuthal velocity associated with the state with the jet near the top. These steady axisymmetric one-vortex states are stable in region I. There are other flow states that are stable in this same region. For example, above the dashed curve SN in figure 23(a), the two-Taylor-vortex state becomes stable and coexists with the one-vortex states. However, the two-vortex and the one-vortex states are well separated in phase space and the experiments and numerics we describe below are focused on the one-vortex state. On increasing  $Re$ , the one-vortex state suffers a Hopf bifurcation that breaks the  $SO(2)$  symmetry and a rotating wave state emerges with azimuthal wave number  $m = 2$ . Figure 23(b) shows an isosurface of axial angular momentum, illustrating the three-dimensional structure of the rotating wave. For slight variations in aspect ratio, the rotating wave may precess either prograde (in region  $\text{II}_+$  above  $H_+$ ) or retrograde (in region  $\text{II}_-$  above  $H_-$ ) with the inner cylinder, and in between a pinning region III is observed. This is observed even with a nominally perfect experimental system, i.e. with the  $SO(2)$  symmetry to within the tolerances in constructing the apparatus. The Hopf bifurcation is supercritical around

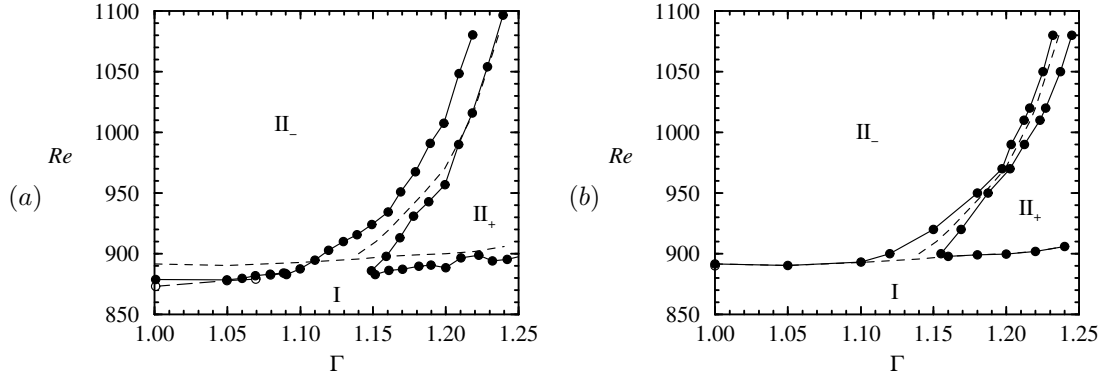


Figure 24: Bifurcation diagrams for the one-cell state from (a) the experimental results of Abshagen *et al.* (2008) with the natural imperfections of their system, and (b) the numerical results of Pacheco *et al.* (2011) with a tilt of  $0.1^\circ$  on the upper lid. The dotted curve in both is the numerically determined Hopf curve with zero tilt.

the region where the precession frequency changes sign. However, for smaller aspect ratios the Hopf bifurcation becomes subcritical at the Bautin point Ba in figure 23(a).

Various different experiments in this regime have been conducted in the nominally perfect system, as well as with a small tilt of an endwall (Pfister *et al.*, 1988, 1991, 1992; Abshagen *et al.*, 2008). Figure 24(a) shows a bifurcation diagram from the laboratory experiments of Abshagen *et al.* (2008). These experiments show that without an imposed tilt, the natural imperfections of the system produce a measurable pinning region, and that the additional tilting of one endwall increases the extent of the pinning region. Tilt angles of the order of  $0.1^\circ$  are necessary for the tilt to dominate over the natural imperfections. Figure 24(b) shows a bifurcation diagram from the numerical results of Pacheco *et al.* (2011), including a tilt of one of the endwalls of about  $0.1^\circ$ , showing very good agreement with the experimental results. Included in figures 24(a) and (b) is the numerically computed bifurcation diagram in the perfect system, shown as dotted curves. The effects of imperfections are seen to be only important in the parameter range where the Hopf frequency is close to zero and a pinning region appears. It is bounded by infinite-period bifurcations of limit cycles. The correspondence between these results and the normal form theory described in this paper is excellent, strongly suggesting that the general remarks on pinning extracted from the analysis of the five particular cases are indeed realized both experimentally and numerically. These two studies (Abshagen *et al.*, 2008; Pacheco *et al.*, 2011) are the only cases we know of where quantitative data about the pinning region are available. Yet, even in these cases the dynamics close to the intersection of the Hopf curve with the pinning region, that according to our analysis should include complicated bifurcational processes, has not been explored either numerically or experimentally. This is a very interesting problem that deserves further exploration.

## 8.2 Pinning in rotating Rayleigh-Bénard convection

Up to now, we have considered the zero-frequency Hopf problem in the context of a supercritical Hopf bifurcation. However, in the Taylor-Couette example discussed in the previous section, the zero frequency occurs quite close to a Bautin bifurcation, at which the Hopf bifurcation switches from being supercritical to subcritical, and a natural question is what are the consequences of the zero-frequency occurring on a subcritical Hopf bifurcation. The normal form theory for the behavior local to the Hopf bifurcation carries over by changing the direction of time and the sign of the parameters  $\mu$  and  $\nu$  as discussed before, but then both the limit cycle and the pinned state are unstable and not observable in a physical experiment or direct numerical simulation. The limit cycle becomes observable as it undergoes a saddle-node of limit cycles (a cyclic fold) bifurcation at the fold associated with the Bautin bifurcation (see a schematic in figure 25a), and we expect that the pinned state does likewise with a saddle-node of fixed points bifurcation along the same fold.

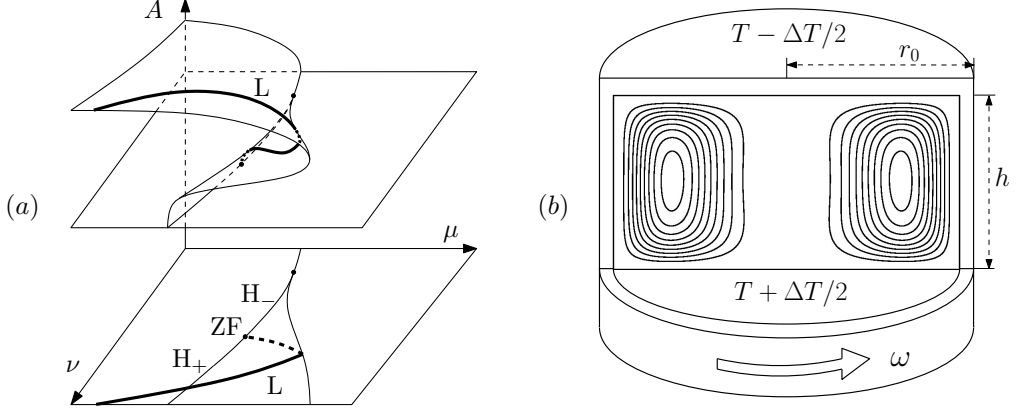


Figure 25: (a) Schematic of the Bautin bifurcation, including the path  $L$  of frequency zero bifurcated states in the  $SO(2)$  perfect system, shown in a two parameter space  $(\mu, \nu)$  with  $A$  a global measure of the solution. Also shown is the projection of the saddle-node surface on parameter space. (b) Schematic of the rotating convection apparatus, with the streamlines of the basic state shown in the inset.

In this subsection, we identify a rotating convection problem where precisely this occurs (Marques *et al.*, 2007; Lopez & Marques, 2009), and conduct new numerical simulations by introducing an  $SO(2)$  symmetry-breaking bifurcation that produces a pinning region on the upper branch of the subcritical Hopf bifurcation.

The rotating convection problem consists of the flow in a circular cylinder of radius  $r_0$  and height  $h$ , rotating at a constant rate  $\omega$  rad/s. The cold top and hot bottom endwalls are maintained at constant temperatures  $T_0 \pm 0.5\Delta T$ , where  $T_0$  is the mean temperature and  $\Delta T$  is the temperature difference between the endwalls. The sidewall has zero heat flux. Figure 25(b) shows a schematic of the flow configuration.

Using the Boussinesq approximation that all fluid properties are constant except for the density in the gravitational and centrifugal buoyancy terms, and using  $h$  as the length scale,  $h^2/\kappa$  as the time scale, and  $\Delta T$  as the temperature scale, the governing equations written in the rotating frame of reference are:

$$(\partial_t + \mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \sigma \nabla^2 \mathbf{u} + \sigma Ra \Theta \hat{z} + 2\sigma \Omega \mathbf{u} \times \hat{z} - \frac{\sigma Fr Ra}{\gamma} (\Theta - z) \mathbf{r}, \quad (42)$$

$$(\partial_t + \mathbf{u} \cdot \nabla) \Theta = w + \nabla^2 \Theta, \quad \nabla \cdot \mathbf{u} = 0, \quad (43)$$

where  $\mathbf{u} = (u, v, w)$  is the velocity field in cylindrical coordinates  $(r, \theta, z)$  in the rotating frame,  $p$  is the kinematic pressure (including gravitational and centrifugal contributions),  $\hat{z}$  the unit vector in the vertical direction  $z$ , and  $\mathbf{r}$  is the radial vector in cylindrical coordinates. Instead of the non-dimensional temperature  $T$ , we have used the temperature deviation  $\Theta$  with respect to the conductive profile,  $T = T_0/\Delta T - z + \Theta$ , as is customary in many thermal convection studies.

There are five non-dimensional independent parameters:

$$\begin{aligned} \text{the Rayleigh number} & \quad Ra = \alpha g h^3 \Delta T / (\kappa \nu), \\ \text{the Froude number} & \quad Fr = \omega^2 r_0 / g, \\ \text{the Coriolis number} & \quad \Omega = \omega h^2 / \nu, \\ \text{the Prandtl number} & \quad \sigma = \nu / \kappa, \\ \text{the aspect ratio} & \quad \gamma = r_0 / h, \end{aligned} \quad (44)$$

where  $\alpha$  is the coefficient of volume expansion,  $g$  is the gravitational acceleration,  $\kappa$  is the thermal diffusivity, and  $\nu$  is the kinematic viscosity.



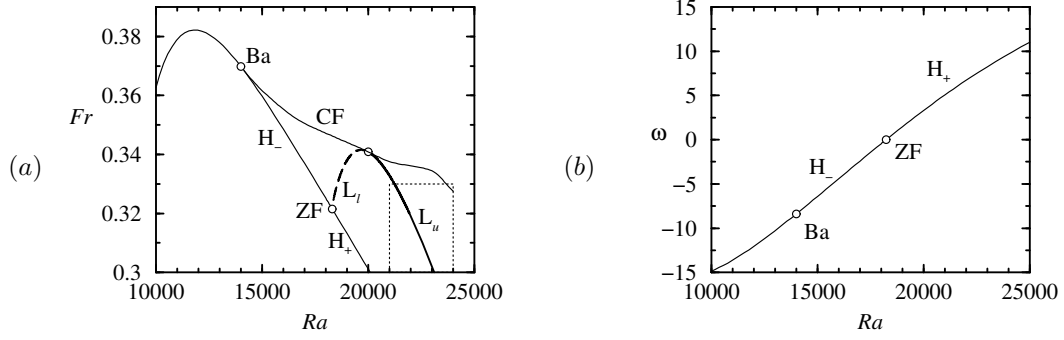


Figure 26: (a) Bifurcation curves for  $\Omega = 100$ ,  $\sigma = 7$  and  $\gamma = 1$ , where  $H_{\pm}$  are the segments of the Hopf bifurcation with negative and positive frequency, the Bautin point Ba is where the Hopf bifurcation switches from super- to subcritical and the cyclic-fold bifurcation curve CF emerges. The lines  $L_l$  and  $L_u$  are the loci where the rotating wave has zero frequency on the lower and upper branches of the cyclic fold. The rectangle around  $L_u$  corresponds to figure 27a. (b) The frequency along the Hopf bifurcation  $H_{\pm}$  with the Bautin point and the point ZF, where the sense of precession changes, marked as open symbols.

The boundary conditions for  $\mathbf{u}$  and  $\Theta$  are:

$$r = \gamma : \quad \Theta_r = u = v = w = 0, \quad (45)$$

$$z = \pm 1/2 : \quad \Theta = u = v = w = 0. \quad (46)$$

For any  $Fr \neq 0$ , the system is not invariant to the so-called Boussinesq symmetry corresponding to invariance to a reflection  $K_z$  about the half-height  $z = 0$ , whose action is  $K_z(u, v, w, \Theta, p)(r, \theta, z) = (u, v, -w, -\Theta, p)(r, \theta, -z)$ . The system is only invariant under rotations about the axis of the cylinder, the  $SO(2)$  symmetry.

The governing equations have been solved using a second-order time-splitting method combined with a pseudo-spectral method for the spatial discretization, utilizing a Galerkin–Fourier expansion in the azimuthal coordinate  $\theta$  and Chebyshev collocation in  $r$  and  $z$ . The details are presented in Mercader *et al.* (2010). We have used  $n_r = 36$ ,  $n_\theta = 40$  and  $n_z = 64$  spectral modes in  $r$ ,  $\theta$  and  $z$  and a time-step  $dt = 2 \times 10^{-5}$  thermal time units in all computations. We have checked the spectral convergence of the code using the infinity norm of the spectral coefficients of the computed solutions. The trailing coefficients of the spectral expansions are at least five orders of magnitude smaller than the leading coefficients. In order to compute the zero-frequency line  $L$  in the subcritical region of the Bautin bifurcation, where the fixed points and limit cycles involved are unstable, we have used arclength continuation methods for fixed points and for rotating waves adapted to our spectral codes (Sanchez *et al.*, 2002; Mercader *et al.*, 2006).

Figure 26(a) shows the parameter region of interest in this convection problem. In the region of high Froude number (region I) we have a stable steady solution, consisting of a single axisymmetric convective roll where the warm fluid moves upwards close to the axis (due to the rotation of the container), and returns along the sidewall, as illustrated in figure 25(b); the inset shows streamlines of the flow in this base state, that is  $SO(2)$ -equivariant with respect to rotations about the cylinder axis. This base state loses stability when the Froude number  $Fr$  decreases, in a Hopf bifurcation along the curves  $H_{\pm}$ . The bifurcation is supercritical for  $Ra < 14157$  and subcritical for higher  $Ra$ ; the change from supercritical to subcritical happens at the codimension-two Bautin bifurcation point Ba, at  $(Ra, Fr) \approx (14157, 0.3684)$ . The bifurcated limit cycle, a rotating wave with azimuthal wave number  $m = 3$ , is unstable, but becomes stable at the cyclic fold curve CF (a saddle-node bifurcation of limit cycles). This curve CF originates at the Bautin point Ba. There are other flow states that are stable in this same region (Lopez & Marques, 2009); these additional states are well separated in phase space and the numerics we describe below are focused on the base state and the  $m = 3$  bifurcated rotating wave.

Figure 26(b) shows the computed frequency of the limit cycle along the Hopf bifurcation curve.

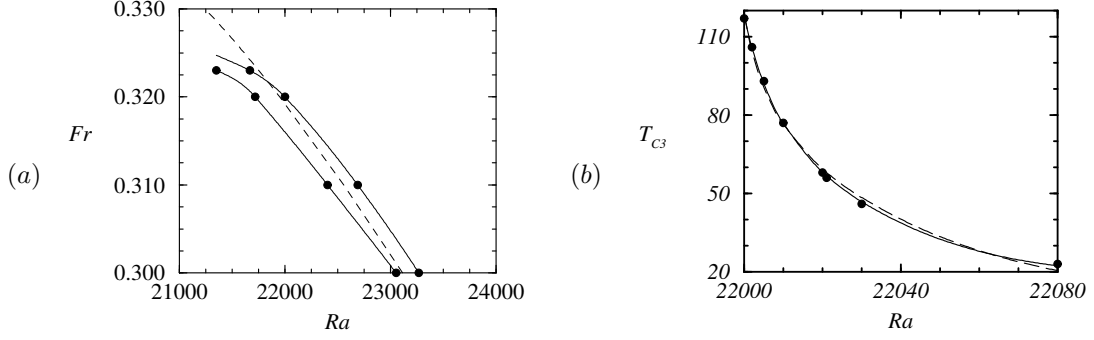


Figure 27: (a) SNIC bifurcation curves in  $(Ra, Fr)$  space bounding the pinning region for  $\Omega = 100$ ,  $\sigma = 7$  and  $\gamma = 1$ . The region shown is the rectangle in figure 26a, and the dashed line is the  $L_u$  curve in the perfect case. (b) The period of the rotating wave C3 as it approaches the SNIC bifurcations at  $Fr = 0.32$  and other parameters as in (a).

This frequency is negative along  $H_-$  and positive along  $H_+$ , and is zero at the ZF (zero frequency) point. At this point we have precisely the scenario discussed in the present paper: a flow (the base state) with  $SO(2)$  symmetry undergoing a Hopf bifurcation that has zero frequency at that point. Figure 26(a) also includes the line L in parameter space where the frequency of the bifurcated states is zero. This curve has been computed using continuation methods since the zero-frequency state is unstable in the lower part ( $L_l$ ) of the saddle-node CF, and therefore cannot be obtained via time evolution. The zero frequency state becomes stable upon crossing the saddle-node curve CF and moving to the upper part  $L_u$  of the saddle-node CF, and becomes observable both experimentally and by numerical simulations advancing the Navier-Stokes equations in time.

In order to break the  $SO(2)$  symmetry and see if a pinning region appears, an imperfection has been introduced, in the form of an imposed linear profile of temperature at the top lid:

$$\Theta(r, \theta, z) = \epsilon r \cos \theta \quad \text{at} \quad z = 1/2, \quad (47)$$

where  $\epsilon$  is a measure of the symmetry breaking. This term completely breaks the rotational symmetry of the governing equations, and no symmetry remains. Figure 27(a) shows that the line L becomes a band of pinned solutions, steady solutions with frequency zero, as predicted by the normal form theory presented in this paper. We can also check the nature of the bifurcation taking place at the boundary of the pinning region. Figure 27(b) shows the variation of the period of the limit cycle approaching the pinning region. It is an infinite period bifurcation, and the square root fit (shown in the figure) works better than the logarithmic fit. We estimate that the bifurcation is a SNIC bifurcation, as the normal form theory presented predicts it should be sufficiently far from the zero frequency point ZF.

Figure 28 shows contours of the temperature at a horizontal section at mid height ( $z = 0$ ) for the symmetric system ( $\epsilon = 0$ ) in figure 28(a), and for the system with an imperfection  $\epsilon = 0.05$ , corresponding to a maximum variation of temperature of  $5\% \Delta T$  at the top lid, in figure 28(b). The parameter values for both snapshots are  $Ra = 21950$  and  $Fr = 0.32$ , inside the pinning region in figure 27(a). The pinned solution 28(b) has broken the  $SO(2)$  symmetries, is a steady solution, and we can see that one of the three arms of the solution is closer to the wall than the other two. The attachment of the solution to the side wall, due to the imperfection at the top lid, results in the pinning phenomenon.

## 9 Summary and conclusions

The aim of this paper has been to provide a general dynamical systems description of the pinning phenomenon which is observed in systems possessing two ingredients: slowly traveling or rotating waves and imperfections. The description boils down to the unfolding of a Hopf bifurcation in an

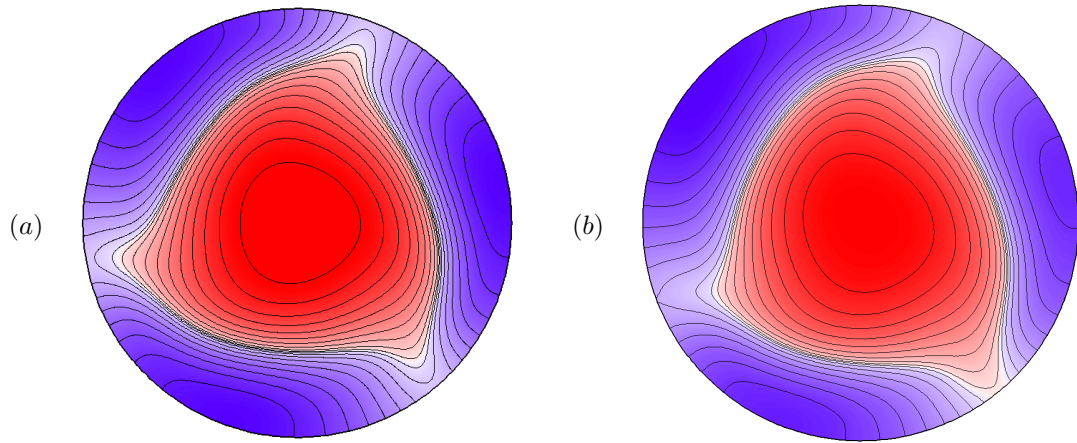


Figure 28: Temperature contours at mid height ( $z = 0$ ) for  $\Omega = 100$ ,  $\sigma = 7$ ,  $\gamma = 1$ ,  $Ra = 21950$  and  $Fr = 0.32$ . (a) is the symmetric solution without imperfection ( $\epsilon = 0$ ), and (b) is a pinned solution with an imperfection  $\epsilon = 0.05$ . There are 20 quadratically spaced contours in the interval  $T \in [-0.31, 0.31]$ , with blue (red) for the cold (warm) fluid.

$SO(2)$  equivariant system about the point where the Hopf frequency is zero. This turns out to be a very complicated problem due to the degeneracies involved, but by considering all of the low-order ways in which  $SO(2)$  symmetry may be broken near a zero frequency Hopf bifurcation, we can identify a number of general features which are common to all scenarios, and hence can be expected to be found in practice. These are that the curve of zero frequency splits into a region in parameter space of finite width that scales with the strength of the imperfection, and this region is delimited by SNIC bifurcations. In the very small neighborhood of the zero frequency Hopf bifurcation point, where the SNIC curves and the Hopf curve approach each other, the dynamics is extremely complicated, consisting in a multitude of codimension-two local bifurcations and global bifurcations. The details depend on the particulars of the imperfection, but all of these complications are very localized and are not resolvable in any practical sense. We provide two examples in canonical fluid dynamics to illustrate both the pinning phenomenon and the use of the theory to describe it. These are a Taylor-Couette flow in which the Hopf bifurcation is supercritical and a rotating Rayleigh-Bénard flow where the Hopf bifurcation is subcritical.

## Acknowledgments

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## A Notation and description of bifurcations

Codimension-one bifurcations	
Name	Description
$\text{SN}_{\pm,0}$	Saddle-node (also called fold) bifurcations
$\text{H}_{\pm,0}$	Hopf bifurcations
$\text{PF}_{\pm}$	Pitchfork bifurcations
CF	Cyclic fold: two limit cycles are born simultaneously
L, $\text{L}_{l,u}$	Limit cycle becoming a family of fixed points
$\text{Hom}_{\pm,0}$ , Hom	Homoclinic collision of a limit cycle with a saddle
$\text{Het}_{\pm,0}$	Heteroclinic collision of a limit cycle with saddles
$\text{SNIC}_{\pm,0}$	Saddle-node appearing on a limit cycle
Glu	Gluing bifurcation – two limit cycles collide with a saddle
Codimension-two bifurcations	
Name	Description
$\text{Cusp}_{\pm}$	Cusp bifurcations
$\text{TB}_{\pm}$ , TB	Takens-Bogdanov bifurcations
$\text{dPF}_{\pm}$	Degenerate pitchfork – zero cubic term
Ba	Bautin bifurcation – degenerate Hopf with zero cubic term
$\text{PfGl}$	Simultaneous gluing Gl and pitchfork PF bifurcations
$\text{CfHom}$	Simultaneous cyclic-fold CF and homoclinic collision Hom
$\text{CfHet}_{\pm}$	Simultaneous cyclic-fold CF and heteroclinic collision Hom
$\text{SnicHom}_{\pm,0}$	Simultaneous SNIC and homoclinic collision
$\text{SnicHet}_{\pm,0}$	Simultaneous SNIC and heteroclinic collision

## B Symmetry breaking of $SO(2)$ with an $\epsilon$ term: computations

### Fixed points

The fixed points of the normal form (12) are given by  $\dot{r} = \dot{\phi} = 0$ , i.e.

$$\left. \begin{aligned} \cos \phi &= r(ar^2 - \mu), \\ \sin \phi &= r(\nu - br^2), \end{aligned} \right\} \Rightarrow r^2[(\mu - ar^2)^2 + (\nu - br^2)^2] = 1, \quad (48)$$

resulting in the cubic equation  $f(\rho) = \rho^3 - 2(a\mu + b\nu)\rho^2 + (\mu^2 + \nu^2)\rho - 1 = 0$ , where  $\rho = r^2$ . This equation always has a real solution with  $\rho > 0$ , and in some regions in parameter space may have three solutions. The curve separating these behaviors is a curve of saddle-node bifurcations, where a couple of additional fixed points are born. This saddle-node curve is given by  $f(\rho) = f'(\rho) = 0$ ; from these equations we can obtain  $(\mu, \nu)$  as a function of  $\rho$ . In order to describe the curve it is better to use the rotated reference frame  $(u, v)$ , where the  $u$ -axis coincides with the line L, introduced in (13) (see also figure 2(a)). The saddle-node curve is given by

$$(u, v) = \frac{1}{2\rho_2^2} \left( 1 + 2\rho_2^3, \pm \sqrt{4\rho_2^3 - 1} \right), \quad \rho_2 \in (2^{-2/3}, +\infty), \quad (49)$$

where  $\rho_2$  is the double root of the cubic equation  $f(\rho) = 0$ . The third root  $\rho_0$  is given by  $\rho_0\rho_2^2 = 1$ . If there is any point where the three roots coincide (i.e. a cusp bifurcation point, where two saddle-node curves meet), it must satisfy  $f(\rho) = f'(\rho) = f''(\rho) = 0$ . There are two such points  $\text{Cusp}_{\pm}$ , given by  $\rho_2 = 1$  and  $(u, v)_{\text{Cusp}_{\pm}} = (3/2, \pm\sqrt{3}/2)$ , dividing the saddle-node curve into three branches:

$\text{SN}_0$ , joining  $\text{Cusp}_+$  and  $\text{Cusp}_-$ , and unbounded branches  $\text{SN}_+$  and  $\text{SN}_-$  starting at  $\text{Cusp}_+$  and  $\text{Cusp}_-$  respectively, and becoming asymptotic to the line  $L$ . Along  $\text{SN}_0$ ,  $\rho_2 < 1 < \rho_0$ , while along  $\text{SN}_+$  and  $\text{SN}_-$ ,  $\rho_0 < 1 < \rho_2$ . At the cusp points, the three roots coincide and their common value is  $+1$ .

A better parametrization of the saddle-node curve is obtained by introducing  $s = \pm\sqrt{(4\rho_2^3 - 1)/3}$ , so that now  $s \in (-\infty, +\infty)$ ,  $\text{Cusp}_\pm$  corresponds to  $s = \pm 1$  and

$$(u, v) = \frac{(3(1 + s^2), 2\sqrt{3}s)}{[2(1 + 3s^2)]^{2/3}}, \quad \rho_0 = \left(\frac{4}{1 + 3s^2}\right)^{2/3}, \quad \rho_2 = \left(\frac{1 + 3s^2}{4}\right)^{1/3}. \quad (50)$$

## Hopf bifurcations of the fixed points

Using Cartesian coordinates  $z = x + iy$  in (12) we obtain

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \mu & -\nu \\ \nu & \mu \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - (x^2 + y^2) \begin{pmatrix} ax - by \\ bx + ay \end{pmatrix}, \quad (51)$$

where we have set  $\epsilon = 1$ . The Jacobian of the right-hand side of (51) is given by

$$J = \begin{pmatrix} \mu - 3ax^2 - ay^2 + 2bxy & -\nu + bx^2 + 3by^2 - 2axy \\ \nu - 3bx^2 - by^2 - 2axy & \mu - ax^2 - 3ay^2 - 2bxy \end{pmatrix}. \quad (52)$$

The invariants of the Jacobian are given by

$$T = 2(\mu - 2ar^2), \quad D = \mu^2 + \nu^2 - 4(a\mu + b\nu)r^2 + 3r^4. \quad (53)$$

A Hopf bifurcation takes place iff  $T = 0$  and  $D > 0$ . When  $T = 0$ ,  $4a^2D = 4(a\nu - b\mu)^2 - \mu^2$ . The fixed points satisfying  $T = 0$  are given by  $f(\rho) = 0$  and  $\mu = 2a\rho$ , resulting in the curve  $T$  in parameter space

$$4a\mu\nu(a\nu - b\mu) = 8a^3 - \mu^3, \quad (54)$$

that can be parametrized as

$$(\mu, \nu) = a^{1/3}(1 - s^2)^{1/3} \left( 2, \frac{b}{a} + \frac{s}{\sqrt{1 - s^2}} \right), \quad s \in (-1, +1). \quad (55)$$

For  $s \rightarrow \pm 1$ ,  $\mu = 0$  and  $\nu \rightarrow \pm\infty$ , and the curve is asymptotic to the  $\mu = 0$  axis, the Hopf curve for  $\epsilon = 0$ . Along the  $T$  curve, the determinant  $D$  is given by

$$D = a^{2/3}(1 - s^2)^{-1/3} \left( 2s^2 - 1 - 2\frac{b}{a}s\sqrt{1 - s^2} \right), \quad (56)$$

resulting in two Hopf bifurcation curves (when  $D > 0$ ):

$$\text{H}_- : s \in \left( -1, -\sqrt{(1 - b)/2} \right), \quad \text{H}_+ : s \in \left( \sqrt{(1 + b)/2}, +1 \right). \quad (57)$$

The end points of these curves have  $T = D = 0$ , and are Takens–Bogdanov bifurcation points  $\text{TB}_\pm$ . They are precisely on the saddle-node curve (50), where both curves are tangent. The coordinates of the four points  $\text{Cusp}_\pm$  and  $\text{TB}_\pm$  are:

$$(\mu, \nu)_{\text{Cusp}_+} = \frac{3}{2} \left( a - \frac{b}{\sqrt{3}}, b + \frac{a}{\sqrt{3}} \right), \quad (\mu, \nu)_{\text{Cusp}_-} = \frac{3}{2} \left( a + \frac{b}{\sqrt{3}}, b - \frac{a}{\sqrt{3}} \right), \quad (58)$$

$$(\mu, \nu)_{\text{TB}_+} = \frac{(2a, 2b + 1)}{(2(1 + b))^{1/3}}, \quad (\mu, \nu)_{\text{TB}_-} = \frac{(2a, 2b - 1)}{(2(1 - b))^{1/3}}. \quad (59)$$

As  $\text{TB}_\pm$  are on the saddle-node curve, its  $\tilde{s}_\pm$  parameter, according to (50), can be computed. The result is  $\tilde{s}_\delta = \delta\sqrt{(1 - \delta b)/3(1 + \delta b)}$ , with  $\delta = \pm'$ . Therefore  $\text{TB}_+ \in \text{SN}_0$ , closer to  $\text{Cusp}_+$  than to  $\text{Cusp}_-$ ;  $\text{TB}_- \in \text{SN}_-$  if  $\alpha_0 < 60^\circ$ ,  $\text{TB}_- \in \text{SN}_0$  when  $\alpha_0 > 60^\circ$  and  $\text{TB}_- = \text{Cusp}_-$  for  $\alpha_0 = 60^\circ$ .

## C Symmetry breaking of $SO(2)$ to $Z_2$ : computations

### Fixed points

The fixed points of the normal form (19) are given by  $\dot{r} = \dot{\phi} = 0$ . One solution is  $r = 0$  (named  $P_0$ ). The other fixed points are the solutions of

$$\left. \begin{aligned} \cos 2\phi &= ar^2 - \mu, \\ \sin 2\phi &= \nu - br^2, \end{aligned} \right\} \Rightarrow (\mu - ar^2)^2 + (\nu - br^2)^2 = 1, \quad (60)$$

resulting in the bi-quadratic equation  $r^4 - 2(a\mu + b\nu)r^2 + \mu^2 + \nu^2 - 1 = 0$ , whose solutions are

$$r_{\pm}^2 = a\mu + b\nu \pm \Delta, \quad e^{2i\phi_{\pm}} = (a\nu - b\mu \mp i\Delta)e^{i\alpha_0} \quad (61)$$

$$\Delta^2 = (a\mu + b\nu)^2 + 1 - \mu^2 - \nu^2 = 1 - (a\nu - b\mu)^2, \quad (62)$$

and every  $\phi_{\pm}$  admits two solutions, differing by  $\pi$  (they are related by the symmetry  $Z_2$ ,  $z \rightarrow -z$  discussed above). Introducing a new phase  $\alpha_1$ ,

$$a\nu - b\mu - i\Delta = e^{i\alpha_1}, \quad (63)$$

where  $\alpha_1 \in [-\pi, 0]$  because  $\Delta > 0$ , we immediately obtain:

$$e^{2i\phi_{\pm}} = e^{i(\alpha_0 \pm \alpha_1)} \Rightarrow \phi_{\pm} = (\alpha_0 \pm \alpha_1)/2, \quad (64)$$

with the other solution being  $(\alpha_0 \pm \alpha_1)/2 + \pi$ ;  $\alpha_1$  is a function of  $(\mu, \nu)$  while  $\alpha_0$  is a fixed constant. We have obtained two pairs of  $Z_2$  symmetric points,  $P_+ = r_+ e^{i\phi_+}$  and  $P_+^* = -r_+ e^{i\phi_+}$ , and  $P_- = r_- e^{i\phi_-}$  and  $P_-^* = -r_- e^{i\phi_-}$ .

### Hopf bifurcations of fixed points

The Jacobian of the right-hand side of (26) is given by

$$J = \begin{pmatrix} \mu + 1 - 3ax^2 - ay^2 + 2bxy & -\nu + bx^2 + 3by^2 - 2axy \\ \nu - 3bx^2 - by^2 - 2axy & \mu - 1 - ax^2 - 3ay^2 - 2bxy \end{pmatrix}. \quad (65)$$

The invariants of the Jacobian are the trace  $T$ , the determinant  $D$  and the discriminant  $Q = T^2 - 4D$ . They are given by

$$T = 2(\mu - 2ar^2), \quad (66)$$

$$D = \mu^2 + \nu^2 - 1 - 4(a\mu + b\nu)r^2 + 3r^4 + 2(a(x^2 - y^2) - 2bxy), \quad (67)$$

$$Q = 4\left(1 - \nu^2 + 4b\nu r^2 + (1 - 4b^2)r^4 - 2(a(x^2 - y^2) - 2bxy)\right). \quad (68)$$

The eigenvalues of the Jacobian matrix (65) in terms of the invariants are  $\lambda_{\pm} = \frac{1}{2}(T \pm \sqrt{Q})$ . For example, a Hopf bifurcation takes place iff  $T = 0$  and  $Q < 0$ . For the fixed points  $P_s$  and  $P_s^*$ , where  $s = \pm$ , we obtain

$$T(P_s) = T(P_s^*) = 2((b^2 - a^2)\mu - 2ab\nu - 2as\Delta), \quad (69)$$

$$D(P_s) = D(P_s^*) = 4s\Delta r_s^2, \quad (70)$$

$$Q(P_s) = Q(P_s^*) = 4((\mu - 2ar_s^2)^2 - 4s\Delta r_s^2). \quad (71)$$

As a result,  $Q(P_-) = Q(P_-^*) > 0$  and  $P_-$  and  $P_-^*$  never experience a Hopf bifurcation. After some computations,  $T(P_+) = 0$  results in the ellipse

$$\mu^2 - 4ab\mu\nu + 4a^2\nu^2 = 4a^2, \quad (72)$$

centered at the origin, contained between the straight lines  $\text{SN}_+$  and  $\text{SN}_-$  and passing through the points  $(\mu, \nu) = (0, \pm 1)$ , the ends of the horizontal diameter of the circle  $\mu^2 + \nu^2 = 1$ . This ellipse is tangent to  $\text{SN}_+$  and  $\text{SN}_-$  at the points  $(\mu, \nu) = \pm(2b, (b^2 - a^2)/a)$ . The condition  $Q < 0$  is only satisfied on the elliptic arc from  $(\mu, \nu) = (0, -1)$  to  $(2b, (b^2 - a^2)/a)$  with  $\mu > 0$ ; along this arc  $P_+$  and  $P_+^*$  undergo a Hopf bifurcation. We have assumed that  $a$  and  $b$  are both positive. The properties of the ellipse are:

$$\begin{array}{ll} \text{major semiaxis} & (1 - \ell_-)\mu = 2ab\nu, \\ \text{minor semiaxis} & (1 - \ell_-)\nu = -2ab\mu, \end{array} \quad \begin{array}{ll} \text{length} & 2a/\sqrt{\ell_-}, \end{array} \quad (73)$$

$$\begin{array}{ll} \text{major semiaxis} & (1 - \ell_+)\mu = 2ab\nu, \\ \text{minor semiaxis} & (1 - \ell_+)\nu = -2ab\mu, \end{array} \quad \begin{array}{ll} \text{length} & 2a/\sqrt{\ell_+}, \end{array} \quad (74)$$

where  $2\ell_{\pm} = 1 + 4a^2 \pm \sqrt{1 + 8a^2}$ . The eccentricity  $e$  is given by

$$\frac{2}{e^2} = 1 + \frac{1 + 4a^2}{\sqrt{1 + 8a^2}}. \quad (75)$$

## Codimension-two bifurcations of fixed points

The Jacobian evaluated at the three points  $\text{TB}_+$ ,  $\text{TB}_-$  and  $\text{TB}$  is:

$$J(\text{TB}_+) = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad (76)$$

$$J(\text{TB}_-) = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad (77)$$

$$J(\text{TB}) = \begin{pmatrix} 1 & (1+b)/a \\ (b-1)/a & -1 \end{pmatrix}. \quad (78)$$

The three matrices have double-zero eigenvalues and are of rank one, so the three of them correspond to Takens–Bogdanov bifurcations. The state that bifurcates at the  $\text{TB}$  point is  $P_+$ , without any symmetry, so that it is an ordinary Takens–Bogdanov bifurcation, although the  $Z_2$  symmetric state  $P_+^*$  also bifurcates at the same point in parameter space (but removed in phase space) at another ordinary Takens–Bogdanov bifurcation. The state that bifurcates at the  $\text{TB}_{\pm}$  points is  $P_0$ . This state is  $Z_2$  symmetric, and so these are Takens–Bogdanov bifurcations with  $Z_2$  symmetry.

The Jacobian evaluated at the two points  $\text{dPF}_{\pm}$  is

$$J(\text{dPF}_+) = \begin{pmatrix} 1-b & -a \\ a & -1-b \end{pmatrix}, \quad J(\text{dPF}_-) = \begin{pmatrix} b+1 & a \\ -a & b-1 \end{pmatrix}. \quad (79)$$

The corresponding eigenvalues are  $\lambda_+ = -2b$  and  $\lambda_- = 0$  for  $\text{dPF}_+$ , and  $\lambda_+ = 2b$  and  $\lambda_- = 0$  for  $\text{dPF}_-$ . Both points are pitchfork bifurcations, and in order to determine if they are degenerate, their normal form needs to be computed in order to verify that the cubic term is zero. However, since in a degenerate pitchfork bifurcation a curve of saddle-node bifurcations emerges that is tangent to the pitchfork bifurcation curve, from figure 7 it is immediate apparent that both  $\text{dPF}_{\pm}$  are degenerate pitchfork bifurcations.

## D Symmetry breaking of $SO(2)$ with quadratic terms: computations

The three cases (32), (35) and (30) can be dealt with by considering the normal form

$$\begin{aligned} \dot{r} &= r(\mu - ar^2) + r^2 \cos m\phi, \\ \dot{\phi} &= \nu - br^2 + r \sin m\phi, \end{aligned} \quad (80)$$

where  $m = 1$  for the  $\epsilon z^2$  case (§5.3),  $m = -1$  for the  $\epsilon z\bar{z}$  case (§5.2) and  $m = -3$  for the  $\epsilon\bar{z}^2$  case (§5.1). The fixed points, other than the trivial solution  $P_0$  ( $r = 0$ ), in the three cases are given by

the biquadratic equation  $r^4 - 2(a\mu + b\nu + 1/2)r^2 + \mu^2 + \nu^2 = 0$ , with solutions  $P_{\pm}$

$$r_{\pm}^2 = a\mu + b\nu + 1/2 \pm (a\mu + b\nu + 1/4 - (a\nu - b\mu)^2)^{1/2}. \quad (81)$$

The phases  $\phi$  of the  $P_{\pm}$  fixed points can be recovered from

$$\cos m\phi = ar - \mu/r, \quad \sin m\phi = br - \nu/r. \quad (82)$$

For  $m = \pm 1$  the solution is unique; for  $m = 3$  the solutions come in triples, differing by  $2\pi/m$ . It is convenient to use the phase space coordinates adapted to line L, introduced in (13) (see also figure 2(a)). In terms of these coordinates,  $r_{\pm}^2 = u + 1/2 \pm \sqrt{u + 1/4 - v^2}$ , and the fixed points  $P_{\pm}$  exist only in the interior of the parabola  $u = v^2 - 1/4$ , whose axis is the line L. On the parabola, these points are born in saddle-node bifurcations. In order to explore additional bifurcations of these points, we compute the Jacobian matrix of the normal form (80),

$$J = \begin{pmatrix} \mu - 3ar^2 + 2r \cos m\phi & -mr^2 \sin m\phi \\ -2br + \sin m\phi & mr \cos m\phi \end{pmatrix}, \quad (83)$$

whose trace and determinant, for the  $P_{\pm}$  points, are easily computed:

$$\begin{aligned} T(P_{\pm}) &= (m-1)ar_{\pm}^2 - (m+1)\mu, \\ D(P_{\pm}) &= -2m\sqrt{u + 1/4 - v^2} \left( \sqrt{(u + 1/2)^2 - u^2 - v^2} \pm (u + 1/2) \right). \end{aligned} \quad (84)$$

Therefore  $\text{sign } D(P_{\pm}) = \mp \text{sign } m$ , and for  $m > 0$  ( $m < 0$ ) only  $P_+$  ( $P_-$ ) may undergo a Hopf bifurcation.

*The  $\epsilon\bar{z}^2$  case (§5.1).* Here  $m = -3$  and  $T = 2\mu - 4ar^2$ .  $P_-$  is a saddle, but  $P_+$  undergoes a Hopf bifurcation when  $T = 0$ . The condition  $T = 0$  for  $P_+$  gives

$$\sqrt{a\mu + b\nu + 1/4 - (a\nu - b\mu)^2} = -\frac{1}{2a} \left( (a^2 - b^2)\mu + 2ab\mu + a \right) > 0. \quad (85)$$

By squaring and simplifying, we obtain the ellipse  $(b\mu - 2a\nu)^2 + (a\mu - 1)^2 = 1$  which is tangent to the line  $\mu = 0$  at the origin, with its center at  $(\mu, \nu) = (2a, b)/(2a^2)$ , and whose elements are:

$$\text{major semiaxis parallel to } (1 - \ell_-)\mu = 2ab\nu, \quad \text{length } 1/\sqrt{\ell_-}, \quad (86)$$

$$\text{minor semiaxis parallel to } (1 - \ell_-)\nu = -2ab\mu, \quad \text{length } 1/\sqrt{\ell_+}, \quad (87)$$

where  $2\ell_{\pm} = 1 + 4a^2 \pm \sqrt{1 + 8a^2}$ . This ellipse has much in common with the one found in the  $\epsilon\bar{z}$  case, and the eccentricity  $e$  is given by the same expression (75). For  $\alpha_0 > \pi/6$ , the ellipse is located in the interior of the parabola of saddle-nodes, for  $\alpha_0 = \pi/6$  it becomes tangent to the parabola at a single point, and for  $\alpha_0 < \pi/6$  it becomes tangent at the two points

$$\begin{aligned} \mu &= \frac{1}{a} (1 - 2a^2 - sb\sqrt{1 - 4a^2}), \\ \nu &= \frac{1}{2a^2} \sqrt{1 - 4a^2} (b\sqrt{1 - 4a^2} - s(1 - 2a^2)), \quad s = \pm 1. \end{aligned} \quad (88)$$

These are the points  $TB_s$  in figure 15(b). Only the points on the elliptic arc  $H_0$  joining these two points satisfy (85), and along this arc  $P_+$  undergoes a Hopf bifurcation.

*The  $\epsilon z\bar{z}$  case (§5.2).* Here  $m = -1$  and  $T(P_{\pm}) = -2ar^2 < 0$ , so there are no Hopf bifurcations. Moreover,  $D(P_+) > 0$ , so it is always stable and  $D(P_-) < 0$ , so it is a saddle. The only exception is when  $r = 0$ , and this only happens at  $\mu = \nu = 0$ , the degenerate high-codimension point at the origin.

*The  $\epsilon z^2$  case (§5.3).* Here  $m = 1$ , and  $T = -2\mu$  is zero on the line  $\mu = 0$  inside the parabola. On this line  $H_0$ ,  $P_+$  undergoes a Hopf bifurcation, and the points of contact with the parabola have  $D = T = 0$  so they are Takens-Bogdanov bifurcations (see figure 17b). The Hopf and Takens-Bogdanov bifurcations are degenerate, as will be discussed in §D.1.



## D.1 A degenerate Takens-Bogdanov bifurcation

In the  $\epsilon z^2$  case, numerical simulations of the normal form (34) show that the Hopf bifurcation  $H_0$  and the Takens–Bogdanov points  $TB_{\pm}$  are degenerate. This can also be found by direct computation. Let us work out the details for the  $TB_-$  point.

The coordinates of  $TB_-$  in parameter space are  $(\mu, \nu) = (0, -0.5/(1+b)) = (0, 1/4 \cos^2(\alpha/2))$ , where  $P_{\pm}$  are born in a saddle-node bifurcation, and the fixed points are given by

$$r_{\pm}^2 = \frac{1}{2(1+b)} = \frac{1}{4 \cos^2(\alpha/2)}, \quad z_{\pm} = \frac{c+i}{2(1+b)} = \frac{ie^{-i\alpha/2}}{4 \cos^2(\alpha/2)}. \quad (89)$$

In order to obtain the normal form corresponding to the Takens–Bogdanov point, a translation of the origin plus a convenient rescaling of  $z$  and time is made:

$$t = 4\tau \cos^2(\alpha/2), \quad \zeta = 2(z - z_{\pm}) \cos(\alpha/2), \quad \tilde{\mu} + i\tilde{\nu} = 4(\mu + i\nu) \cos^2(\alpha/2). \quad (90)$$

Substituting in (34) results in

$$\dot{\zeta} = i\zeta + ie^{-2i\alpha}\bar{\zeta} + e^{i\alpha/2}\zeta^2 + 2e^{-3i\alpha/2}|\zeta|^2 - ie^{-i\alpha}\zeta|\zeta|^2. \quad (91)$$

In order to obtain the normal form, we introduce the real variables  $(x_1, y_1)$

$$\zeta = (y_1 + 2ix_1)e^{-i\alpha}, \quad (92)$$

so that the linear part of the ODE is transformed into Jordan form, and we obtain

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} &= \begin{pmatrix} y_1 \\ 0 \end{pmatrix} + \cos(\alpha/2) \begin{pmatrix} 2x_1y_1 \\ 4x_1^2 + 3y_1^2 \end{pmatrix} + \sin(\alpha/2) \begin{pmatrix} -2x_1^2 - 3y_1^2/2 \\ 4x_1y_1 \end{pmatrix} \\ &\quad - (4x^2 + y^2) \begin{pmatrix} x \sin \alpha + (y/2) \cos \alpha \\ y \sin \alpha - 2x \cos \alpha \end{pmatrix} \end{aligned} \quad (93)$$

Now we can reduce the quadratic and cubic terms to normal form by an appropriate near-identity quadratic transformation  $(x_1, y_1) \rightarrow (x_2, y_2)$ . Knobloch (1986) gives explicitly the normal form coefficients up to and including third order, in terms of the coefficients of the original ODE (in the form 93); a nice summary is also given in Wiggins (2003, §19.9). Using this explicit transformation, we obtain

$$\left. \begin{aligned} \dot{x}_2 &= y_2 \\ \dot{y}_2 &= 4x_2^2 \cos(\alpha/2) + 16x_2^3 \cos^2(\alpha/2) + O(4) \end{aligned} \right\}, \quad (94)$$

and the  $x_2y_2$  term in the normal form of the Takens–Bogdanov bifurcation is missing, resulting in a degenerate case, the so-called cusp case, of codimension three. The unfolding of this degenerate case has been analyzed in detail in Dumortier *et al.* (1987). Note that the ODE (94) is Hamiltonian at least up to order three, which helps to explain the continuous family of periodic orbits obtained in the interior of the homoclinic loop in figure 17(b).

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